INVESTIGATION OF A NON-LINEAR LEAD COMPENSATING NETWORK FOR USE WITH SECOND ORDER D-C SERVOMECHANISMS

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Submitted in partial fulfillment of
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MASTER OF SCIENCE
IN
ELECTRICAL ENGINEERING

United States Naval Postgraduate School
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ABSTRACT

Approximate derivative compensation has been widely used to compensate both d-c and a-c servomechanisms. RC networks provide one of the simplest methods of achieving such compensation. The nonlinear network was originally proposed for use primarily with a-c carrier type systems. This paper consists of an evaluation of the compensator for use with a basic second order d-c servomechanism. A complete mathematical analysis is presented as well as a study using the phase plane as an illustrative tool. The analysis is verified by experimental data obtained from a comprehensive analog computer study. Design data, collected from the computer evaluation, is presented in the form of normalized design curves. These curves define the areas of possible application for the designer, the expected transient response, and the parameters required to obtain the response. It is shown that the curves are quite useful for higher order systems where a reasonable second order approximation may be made. Simple modifications to the basic circuit are discussed and the writers' evaluation, based on computer study is presented.

The writers wish to express their gratitude to the members of the staff of the Electrical Engineering Department who assisted them; to Dr. M. Pastel for his guidance and encouragement during the initial stages of the work; to Dr. G. Thaler for his advice throughout the evaluation; and particularly to Assistant Prof. Gerba whose suggestions and counsel were of inestimable aid in the preparation of this paper.

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TABLE OF SYMBOLS

α	Nonlinear lead network attenuation factor
α_E	Analog computer scale factor for error
α_{E_1}	Analog computer scale factor for output of amplifier #3
α_F	Analog computer scale factor for output of nonlinear network
α_θ	Analog computer scale factor for position
$\alpha_{\dot{\theta}}$	Analog computer scale factor for velocity
α_t	Analog computer scale factor for time
C_A, C_B	Capacitor of nonlinear lead network
$E, \dot{E}, \ddot{E}$	Error, error velocity, and error acceleration respectively
$E_{01}, \dot{E}_{01}$	Error and error velocity at end of Mode I
$E_{12}, \dot{E}_{12}$	Error and error velocity at end of Mode II
F	Output of nonlinear network
G_c	System transfer function
G_o	Forward loop transfer function
G_u	Generalized transfer function for second order servo system
K	Gain added to system when compensated with nonlinear network
K_v	Velocity error coefficient
N	Slope of phase plane trajectory during coast period
R_A	Resistor of nonlinear network
R_o	Output resistor of nonlinear network
t_{c1}	Total time for Mode I
t_{c2}	Total time for Mode II
t_{c3}	Total time for Mode III
t_s	Transient response settling time to within 5% of final value with nonlinear compensation

t_L	Transient response settling time to within 5% of final value with linear lead compensation
ζ	Damping ratio of uncompensated system
ω_n	Natural frequency
θ_R	Reference input
θ_c	Controlled variable output
R	Step input magnitude
θ_{01}	Value of θ_c at end of Mode I
θ_{12}	Value of θ_c at end of Mode II
τ_A	Discharge time constant for nonlinear network
τ_B	Charging time constant for nonlinear network

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1.0 INTRODUCTION

Since 1950 increased emphasis has been placed on the analysis and synthesis of nonlinear servomechanisms. The introduction of the describing function technique by Kochenberger [3] and the increased use of the phase plane for nonlinear analysis after McDonald published his analysis of a dual mode servomechanism [4] in 1952, have provided the servo engineer with practical tools for handling the simpler nonlinear problems.

Various techniques have been presented in recent years demonstrating introduction of nonlinearities to improve system performance [11,13]. In 1956 Clegg [2] published a paper proposing the use of a nonlinear lead network for insertion in the error path. His doctorate thesis [1] in 1957 amplifies the original paper and evaluates the network for application to a-c carrier type servomechanisms. The specific potential of the circuit for compensating d-c systems was not attempted, but the investigation gave promise that the network would be suitable for such systems. The demonstrated ability of the circuit to significantly improve the transient response of an a-c system inspired the investigation contained in this paper to determine if similar results could be obtained with d-c systems.

An attempt is made to satisfactorily answer the following questions: (1) When and where is the nonlinear compensator applicable? (2) What degree of improvement in the transient response is possible? (3) How does the designer determine the parameters required to give the desired response? A preliminary analysis indicated that exact mathematical expressions are conveniently possible only for second order systems. Therefore the detailed evaluation was limited to compensation of second order systems with the assumption that reasonable extension to higher order systems could be expected. The several design parameters involved are the gain of the

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uncompensated system, the two time constants (or the attenuation ratio and one time constant) of the lead networks, and the additional gain to be supplied when the compensating network is inserted. The first limitation assumed is that the steady state velocity error coefficient [7] of the compensated system be equal to that of the uncompensated system. This eliminates the additional gain requirement as a variable. For linear lead networks the range for reasonable attenuation ratios is relatively limited [14]; therefore, the network was evaluated for ratios of 0.2, 0.1, and 0.05. In addition it was determined that a satisfactory computer evaluation was possible only by using a design by synthesis technique. This required that the desired transient response be specified. The type of response generally required is one as rapid as possible with a very small overshoot. The maximum allowable overshoot for a step input was chosen as five per cent. Design curves that will provide the information to obtain this response should be generally applicable as only small parameter adjustments will be required to obtain the actual response desired. With the above limitations all variables for a specific problem were eliminated with the exception of one time constant for the lead network. For the actual circuit the attenuation ratio fixes the relation between the resistors, leaving only the capacitor magnitude as a variable to optimize the circuit.

The following section contains a quantitative description of the network and how it might be expected to operate. Section 3 is the detailed mathematical analysis giving the exact mathematical relationships for the modes of operation. No practical mathematical design methods are attempted. Section 4 utilizes the phase plane technique to explain the detailed operation of the circuit and will quickly familiarize the reader with the

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modes of operation that may be expected. The techniques for the analog computer study are set forth in Section 5. Detailed design information is contained in Section 6, including design curves for each of the chosen attenuation ratios. These curves give parameters to obtain a response with five per cent overshoot to a step input and indicate the response time. Information is included to allow comparison with an ordinary linear lead network. The design curves essentially answer all of the originally posed questions. Various oscillograms are included to demonstrate the compensation possibilities. The last section discusses the information determined from the experimental data, including the possible extension of the design curves to provide data for compensation of higher order systems.

2.0 DESCRIPTION AND QUALITATIVE ANALYSIS

2.1 The Assumed System

The relation of the compensating network to the system to be compensated is shown in Fig. 2-1. [2]. The uncompensated system considered is a generalized second order system which may be described by the transfer function:

$$G_u = \frac{\omega_n^2}{S(S + 2\zeta\omega_n)} \tag{1}$$

The velocity error coefficient, K_v, may be readily determined from Eq. 1 as $\frac{\omega_n}{2\zeta}$ [7]. For analysis purposes it is assumed throughout the report that K_v must remain constant. The effective damping ratio is dependent on the K_v requirement if the parameter $2\zeta\omega_n$ is considered fixed. For some applications this may be an additional variable which will allow greater flexibility for applying the design data given in Section 6.

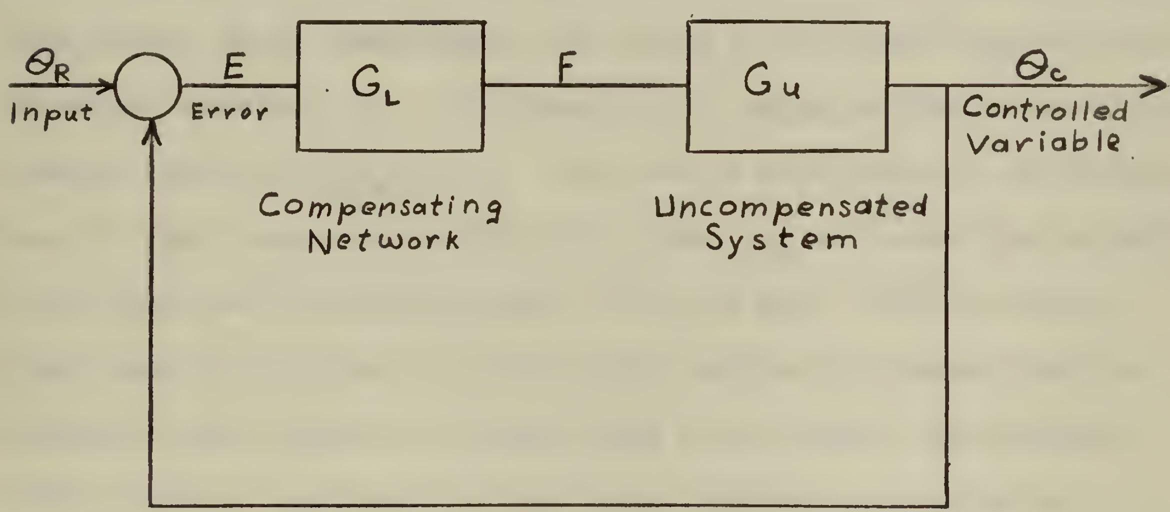

Fig. 2-1 Relation of compensation network to system components.

2.2 The Compensation Network

The basic circuit for the compensation network is as shown in Fig. 2-2 [2] .

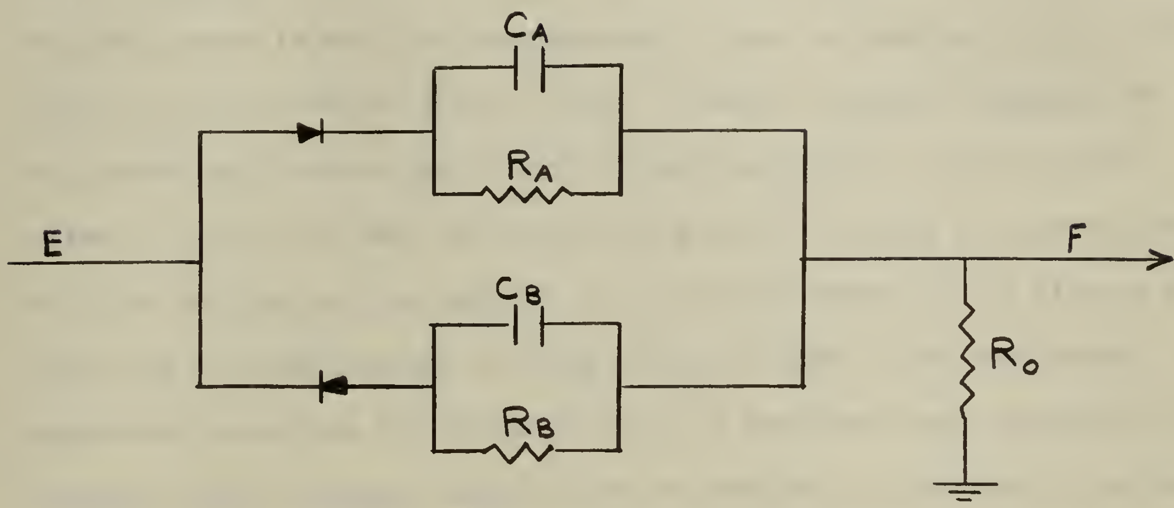

Fig. 2-2 Basic circuit for compensating network.

Consider the qualitative operation of the circuit when placed in the system forward loop. Assume that both capacitors are initially discharged, and there is a positive step function input to the system. For a positive value of the error signal, E, the upper diode conducts. Since the capacitor cannot charge instaneously, the entire error voltage appears across the output resistor, R_o. The capacitor C_A begins to charge with a time constant equal to $R_oC_AR_A/R_A+R_o$. The error is still positive and decreasing. At some instant the voltage drop across C_A will equal the incoming error singal and the output signal, F, will be zero. With the output signal equal to zero and the error signal positive the forward loop has effectively been opened. The upper diode is open due to the capacitor charge, and the lower diode is open since it cannot pass the positive incoming signal. This situation will continue to exist until the error signal goes to zero if the capacitor discharge time constant, R_AC_A, is sufficiently large so that the capacitor charge decreases more slowly than the error

voltage decreases. If $R_A C_A$ is small, and the error is decreasing slowly, the upper diode will alternately open and close to maintain the output voltage nearly zero although some positive signal will be passed. When the output is zero the uncompensated system is coasting toward correspondence with no further driving force. Ideally, inherent damping, the only retarding force active during the coasting period, will cause the system to stop at or near the zero error point. If there is an overshoot the error voltage will go negative and the lower branch of the circuit will conduct in a similar manner to bring the error back to the zero point. For symmetrical operation for either positive or negative input signals the two branches of the nonlinear network must be identical. Accordingly the subscripts for the lower branch will not be used hereafter.

Theoretically this network will produce a response very much like a quasi-optimum relay servo system using "discontinuous damping" proposed by Thaler, et al [5,6] . This scheme assumes that the relay device has some dead zone, and the basic principle is to decelerate the motorload during the dead zone by connecting a dynamic braking resistor across the armature terminals. The nonlinear lead network will have a similar effect. First, the system is accelerated as a third order system with a rapid rise time due to the lead network. When the diode no longer conducts the system will coast as a second order system with short circuit switching, which utilizes maximum dynamic braking. If the optimum cutoff time can be achieved the nonlinear network should improve the response time comparable to what could be achieved with the quasi-optimum relay system.

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3.0 MATHEMATICAL ANALYSIS

3.1 Mode I

The resultant transient response with the nonlinear compensator may be divided into several modes, each of which may be described by linear differential equations. The compensation network block in Fig. 1-1 may be replaced by the actual network to obtain Fig. 3-1.

In Figure 3-1, assume a positive step input of magnitude R. Initially only the upper diode can conduct. Thus the lower branch of the nonlinear network can be removed and the forward loop will appear as in Fig. 3-2.

The usual restrictions are assumed to apply as to coupling, that is, that the input to each block is from an ideal voltage source working into an infinite impedance. The diodes are assumed to be ideal. For some finite period after a step input the nonlinear network will operate exactly as a linear lead network with like parameters. This portion of the response will be referred to as Mode I. The lead network has the transfer function [9].

$$G(S) = \alpha\left(\frac{\tau_A S + 1}{\tau_B S + 1}\right) \qquad \text{where} \qquad \alpha = \frac{R_o}{R_o + R_A}$$

α is the network steady state attenuation factor: τ_A and τ_B are time constants dependent on the circuit parameters.

$$\tau_A = R_A C_A \qquad \text{and} \qquad \tau_B = \alpha \tau_A$$

For a ramp input a steady state attenuation equal to α has been introduced and will result in an increase in velocity lag error. Consequently, additional gain just sufficient to counteract this attenuation will be introduced and included hereafter as an integral part of the compensation network block. Inasmuch as the added gain, K, must equal $1/\alpha$, the complete

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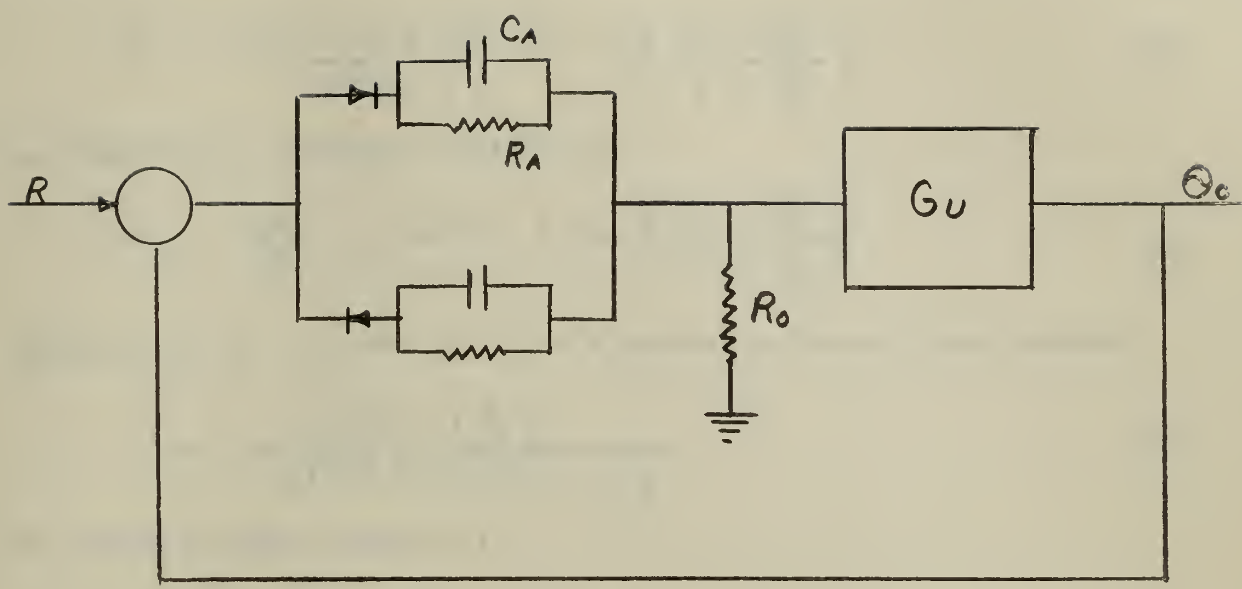


Fig. 3-1 Nonlinear circuit inserted in basic circuit

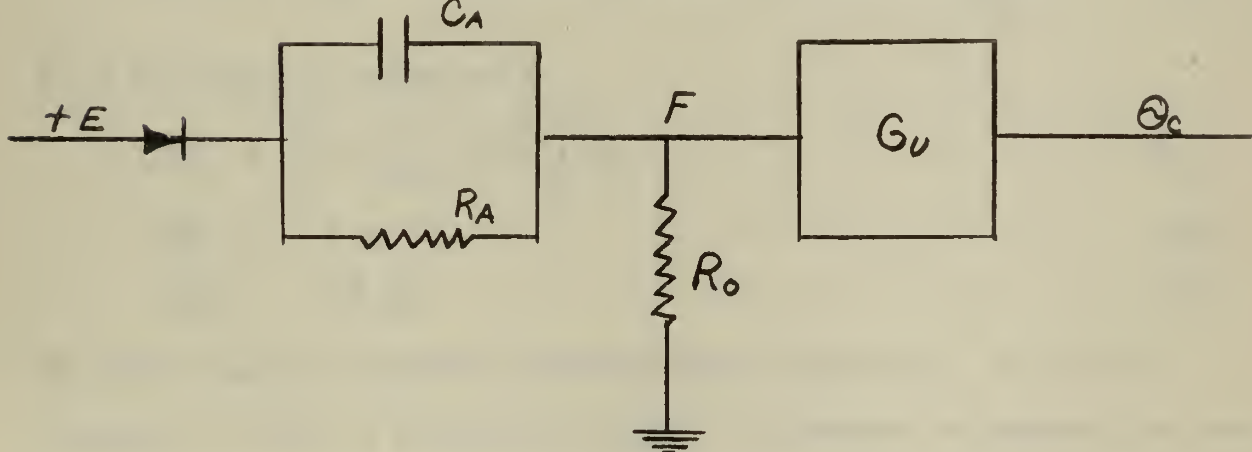
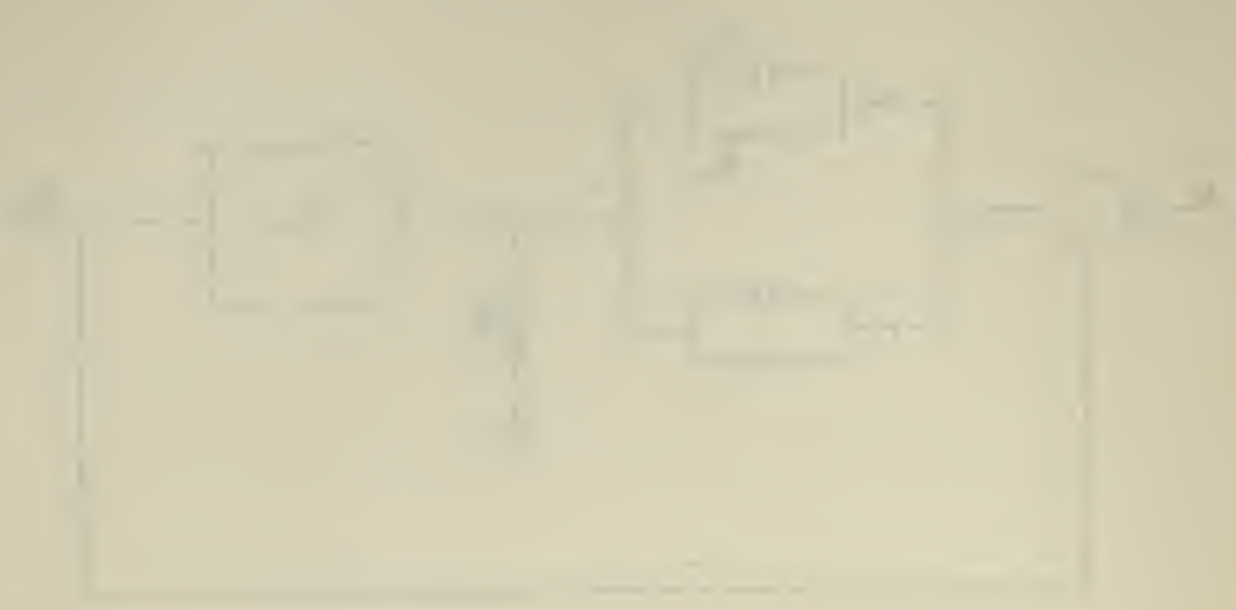


Fig. 3-2 Forward loop for Mode I operation



transfer function for the compensator for Mode I becomes:

$$G_L = K\alpha\left(\frac{\tau_A S + 1}{\alpha \tau_A S + 1}\right) = K\left(\frac{S + \frac{1}{\tau_A}}{S + \frac{K}{\tau_A}}\right) \tag{2}$$

The forward loop transfer function is:

$$G_o = \frac{\theta_c}{E} = G_u G_L = G_u K\left(\frac{S + \frac{1}{\tau_A}}{S + \frac{K}{\tau_A}}\right) \tag{3}$$

Substituting for G_u from Eq. 1 for a generalized second order system:

$$G_o = \frac{K\omega_n^2\left(S + \frac{1}{\tau_A}\right)}{S(S + 2\zeta\omega_n)\left(S + \frac{K}{\tau_A}\right)} \tag{3a}$$

The system transfer function is:

$$G_c = \frac{\theta_c}{\theta_R} = \frac{G_u G_L}{1 + G_u G_L} \tag{4}$$

$$G_c = \frac{K\omega_n^2\left(S + \frac{1}{\tau_A}\right)}{S^3 + \left(\frac{K}{\tau_A} + 2\zeta\omega_n\right)S^2 + \left(2\zeta\omega_n + \omega_n^2\right)KS + \frac{K\omega_n^2}{\tau_A}} \tag{4a}$$

To simplify the notation the following definition is made:

$$D \equiv S^3 + \left(\frac{K}{\tau_A} + 2\zeta\omega_n\right)S^2 + \left(2\zeta\omega_n + \omega_n^2\right)KS + \frac{K\omega_n^2}{\tau_A} \tag{5}$$

For a step input of magnitude R:

$$\theta_c = \left[\frac{K\omega_n^2\left(S + \frac{1}{\tau_A}\right)}{S D}\right] R \tag{6}$$

$$\dot{\theta}_c = S\,\theta_c \tag{6a}$$

$$\ddot{\theta}_c = S^2\,\theta_c \tag{6b}$$

The above equations completely describe Mode I operation. It is now necessary to find its termination point. As discussed in Section 2 the compensation network output signal, F, must go to zero before there is a deviation from linear operation. Therefore, Mode I might also be defined as that part of the response before F goes to zero for the first time.

From Fig. 3-1 and substituting from Eqs. 1 and 6:

$$F = \frac{\Theta_c}{G_u} = \frac{K\left(s+\frac{1}{T_A}\right)\left(s+2\zeta\omega_n\right) R}{D} \tag{7}$$

It is possible to find the inverse transform for F and to solve for the time at which it first becomes zero. Successive approximations are necessary to obtain the solution and the task is too tedious for a design technique, even for a second order system. Eq. 7 shows that the time for F to go to zero is constant and independent of the magnitude of the step input for a given specific system. The cutoff time for Mode I is defined as t_{c_1}. If the inverse transforms for Θ_c and $\dot{\Theta}_c$ (Eqs. 6 and 6a) are found and evaluated at t_{c_1} these values represent the initial conditions for Mode II operations. Since Mode I is linear these values are directly proportional to R. They are defined as:

$$\Theta_c(t_{c_1}) \equiv \Theta_{01} \propto R \tag{8}$$

$$\dot{\Theta}_c(t_{c_1}) \equiv \dot{\Theta}_{01} \propto R \tag{8a}$$

3.2 Mode II

Mode II is defined as the system response from the time, t_{c_1}, when F first goes to zero until such time as the error, E, goes to zero. Assume that after F goes to zero it will remain zero through Mode II operation. As discussed in Section 2 this is one possibility--the other being that an additional positive signal may be transmitted. If F is equal to zero the system must effectively coast. The differential equation describing the coast for a generalized second order system is:

$$\ddot{\Theta}_c + 2\zeta\omega_n\dot{\Theta} = 0 \tag{9}$$

The solution must take into account the initial conditions as defined by Eqs. 8 and 8a. The defining equations for Mode II are:

$$\theta_{c\,III} = \left[\theta_{o1} + \frac{\dot{\theta}_{o1}}{2\zeta\omega_n}\left(1 - \epsilon^{-2\zeta\omega_n t}\right)\right] \tag{9a}$$

$$\dot{\theta}_{c\,III} = \dot{\theta}_{o1}\,\epsilon^{-2\zeta\omega_n t} \tag{9b}$$

$$\ddot{\theta}_{c\,III} = -2\zeta\omega_n\,\dot{\theta}_{o1}\,\epsilon^{-2\zeta\omega_n t} \tag{9c}$$

From Eqs. 8 and 8a it is apparent that the Mode II response is also proportional to the magnitude of the step input. Eqs. 9a, b, c assume time equal to zero at t_{c1}, which must be taken into account for a calculation of total response time. The time for termination of the coast is when error equals zero. That value of time is defined as t_{c2}

Since: $\theta_c = \theta_R - E = R - E$

From Eq. 9a when E = 0:

$$\frac{2\zeta\omega_n}{\dot{\theta}_{o1}}\left(R - \theta_{o1}\right) = 1 - \epsilon^{-2\zeta\omega_n t_{c2}}$$

$$\epsilon^{2\zeta\omega_n t_{c2}} = \frac{\dot{\theta}_{o1}}{\dot{\theta}_{o1} - 2\zeta\omega_n\left(R - \theta_{o1}\right)} \tag{10}$$

$$t_{c2} = \frac{1}{2\zeta\omega_n}\ln\left[\frac{\dot{\theta}_{o1}}{\dot{\theta}_{o1} - 2\zeta\omega_n\left(R - \theta_{o1}\right)}\right] \tag{10a}$$

If error is to ever reach zero in this mode the limiting possibility may be found from Eq. 9a by letting time become infinite. If position is to be equal to or greater than the step input:

$\theta_{o1} + \dfrac{\dot{\theta}_{o1}}{2\zeta\omega_n} \geq R$ but $\theta_{o1} \equiv R - E_{o1}$ where E_{o1} is error at t_{c1}

so that: $-E_{o1} + \dfrac{\dot{\theta}_{o1}}{2\zeta\omega_n} \geq 0$

$$\frac{\dot{\theta}_{o1}}{E_{o1}} \geq 2\zeta\omega_n \leq \frac{\dot{\theta}_{o1}}{(R - \theta_{o1})} \tag{11}$$

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Eq. 11 establishes the required relationship of Θ_{o_1} to $\dot{\Theta}_{o_1}$ for the error to go to zero during the coast period. If the equal sign prevails, the response should appear as critically damped. The time for this response is infinite; however, the value of the system constant $2\zeta\omega_n$ will determine the time for the response to arrive within acceptable limits. For practical systems a small overshoot can normally be tolerated and it will be desirable for the ratio $\frac{\dot{\Theta}_{o_1}}{E_{o_1}}$ to be only slightly greater than $2\zeta\omega_n$. It is quite possible that although F goes to zero the conditions of Eq. 11 are not fullfilled. Physically this means that the capacitor will be discharging faster than the error is decreasing. Eq. 9 will be valid only momentarily after which a positive signal will again be passed by the upper diode. This will again drive F to zero and the sequence will repeat. The response then will be somewhat similar to a rapidly chattering relay. This situation must result in a longer settling time than if the conditions of Eq. 11 are met; therefore, this situation is to be avoided in design unless it is required that the system not overshoot.

Although the values for Eq. 10 may be obtained by analysis of Mode I, this is not very practical as mentioned previously. Eq. 9a contains only one time constant, namely $1/2\zeta\omega_n$. This time constant determines the rate at which the error decreases. The time constant τ_A determines the rate at which the capacitor discharges. A reasonable assumption then is that τ_A must be larger than $1/2\zeta\omega_n$ for Eqs. 9 and 11 to be valid through the range established for Mode II operation. This relation then sets an approximate lower limit for τ_A which is valuable for design purposes.

Thus:
$$\tau_A > \frac{1}{2\zeta\omega_n} \qquad (12)$$

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The upper limit for τ_A presents more of a problem to determine and will be discussed later.

If a system is designed so that the velocity at the end of Mode II (when error is zero) is small, the percent overshoot will be small, since for a negative error the lower diode will conduct and a reverse torque be applied. However, the next mode will be examined prior to making any final conclusions.

3.3 Mode III

Mode III is defined as that portion of the transient response from the time that error first goes to zero until the velocity goes to zero. The termination of Mode III will then define the maximum overshoot. Fig. 3-2 no longer applies in this mode. Error is now negative and the upper diode will not pass the signal. The lower branch will now conduct and the forward loop will appear as in Fig. 3-3 with the capacitor initially uncharged.

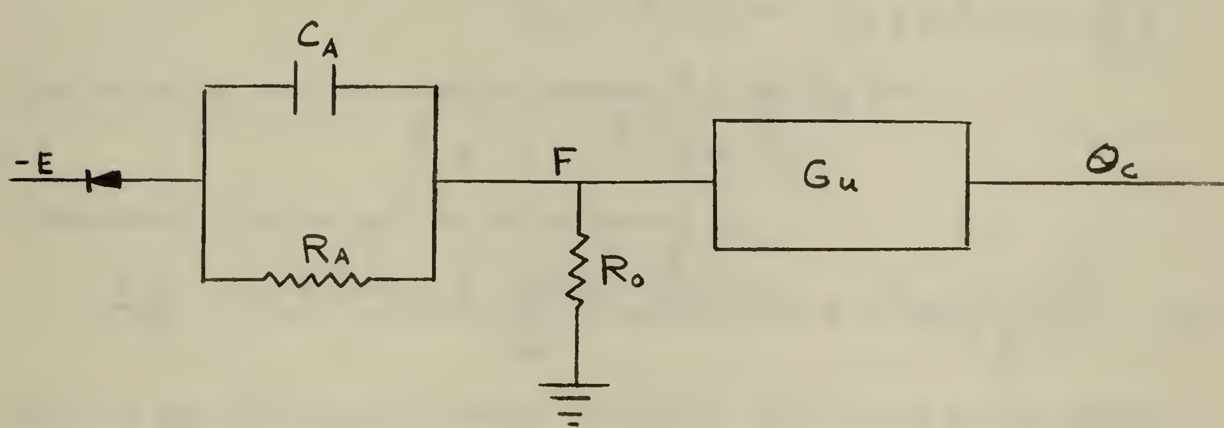

Fig. 3-3 Forward loop for Mode II operation.

The operation must be quite similar to that in Mode I with only the initial conditions changed. Once t_{c2} has been determined Eqs. 9a, 9b, and 9c may be evaluated to give θ_{i2}, $\dot{\theta}_{i2}$ and $\ddot{\theta}_{i2}$ which are assumed to be the initial conditions for Mode III. The error is zero, therefore θ_{i2}

necessarily must be equal to the value of the original step input, R. It will simplify the problem to translate coordinates to eliminate the one initial condition. This amounts to thinking in terms of error rather than position since $E = R - \theta_c$. As for Mode II the initial conditions for Mode III are in direct proportion to the magnitude of the original step input. Eq. 3a is valid for Mode III if modified by the initial conditions. In differential form Eq. 3a becomes:

$$\dddot{\theta}_c + \left(2\zeta\omega_n + \frac{K}{\tau_A}\right)\ddot{\theta}_c + \left(\frac{2\zeta\omega_n K}{\tau_A}\right)\dot{\theta}_c = K\omega_n^2\dot{E} + \frac{K\omega_n^2}{\tau_A}E$$

For a step input: $\theta_c = R - E \; ; \quad \dot{\theta}_c = -\dot{E} \; ; \quad \ddot{\theta}_c = -\ddot{E}$

$$\dddot{E} + \left(2\zeta\omega_n + \frac{K}{\tau_A}\right)\ddot{E} + \left(\frac{2\zeta\omega_n}{\tau_A} + \omega_n^2\right)K\dot{E} + \frac{K\omega_n^2 E}{\tau_A} = 0$$

Transforming by Laplace and remembering that initially E is equal to zero:

$$E(s)\left[s^3 + \left(2\zeta\omega_n + \frac{K}{\tau_A}\right)s^2 + \left(\frac{2\zeta\omega_n}{\tau_A} + \omega_n^2\right)Ks + \frac{K\omega_n^2}{\tau_A}\right] =$$
$$\ddot{E}_{12} + S\dot{E}_{12} + \dot{E}_{12}\left(2\zeta\omega_n + \frac{K}{\tau_A}\right)$$

Eqs 9b and 9c give the relation between $\dot{E}_{12}$ and $\ddot{E}_{12}$ as:

$$\ddot{E}_{12} = -2\zeta\omega_n\dot{E}_{12}$$

Using this relation and the definition of Eq. 5:

$$E_{III} = \frac{\left(-2\zeta\omega_n + S + 2\zeta\omega_n + \frac{K}{\tau_A}\right)\dot{E}_{12}}{D} = \frac{\dot{E}_{12}\left(s + \frac{K}{\tau_A}\right)}{D} \qquad (13)$$

Eqs. 9b and 10 provide the means to express $\dot{E}_{12}$ in terms of the initial conditions for Mode II since $\dot{\theta}_{12}$ (and consequently $-\dot{E}_{12}$) occurs at time t_{c2}. These values were found to be proportional to R (as summarized by Eqs. 8 and 8a). It follows then that E_{III} must likewise be proportional to R, the magnitude of the original step input. The apparent conclusion from this is that once a system is designed for the desired maximum percentage overshoot this percentage will remain nearly constant for a wide

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range of inputs. The transient response for E_{III} may be found from the inverse transform of Eq. 13 and differentiated to obtain error velocity as a function of time. The time that $\dot{E}_{III}$ first becomes zero, t_3, can be found and error evaluated at that time. The value obtained will denote the magnitude of the first overshoot and conclude Mode III operation as defined. At this point the operation of Mode I will repeat, only now there will be a much smaller input which will be negative and the lower diode will continue to conduct. Eq. 7 will not be entirely valid, but for a compensated system would give only a small error; moreover, for small peak overshoots operation past Mode III has little practical significance.

3.4 Limitations

An approximate lower limit for τ_A is established in Section 3.2 above and given by Eq. 12. Conversely this is the upper limit for $1/\tau_A$ and since physically this quanity must be positive:

$$0 < \frac{1}{\tau_A} < 2\zeta\omega_n \tag{14}$$

Using this relation and the one given by Eq. 11 it is possible to distinguish some approximate regions of applicability.

Consider first the case as $1/\tau_A$ approaches zero. Actually this amounts to a simple increase of system gain since as $\tau_A \to 0$ this would physically mean a short circuit of R_A. The system transfer function becomes from Eq. 4a:

$$G_c = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + K\omega_n^2} \tag{15}$$

The roots of the characteristic equation are:

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - K}$$

Since $\zeta \leq 1$ and $K > 1$ $\quad s = -\zeta\omega_n \pm j\,\omega_n\sqrt{\zeta^2 - K}$

Assuming a step input and transforming:

$$\Theta_c = R - R\epsilon^{-\zeta\omega_n t}\left[\cos\omega_n\sqrt{K-\zeta^2}\,t + \frac{\zeta}{\sqrt{K-\zeta^2}}\sin\omega_n\sqrt{K-\zeta^2}\,t\right] \tag{16}$$

Differentiating twice to obtain $\dot{\Theta}_c$ and $\ddot{\Theta}_c$:

$$\dot{\Theta}_c = \frac{K}{\sqrt{K-\zeta^2}}\,\epsilon^{-\zeta\omega_n t}\sin\omega_n\sqrt{K-\zeta^2}\,t \tag{17}$$

$$\ddot{\Theta} = \frac{K}{\sqrt{K-\zeta^2}}\,\epsilon^{-\zeta\omega_n t}\left[\omega_n\sqrt{K-\zeta^2}\cos\omega_n\sqrt{K-\zeta^2}\,t - \zeta\omega_n\sin\omega_n\sqrt{K-\zeta^2}\,t\right] \tag{18}$$

From Fig. 3-1 it may be seen that an initial step input of error will cause a large system acceleration. As the capacitor charges, the acceleration will decrease rapidly and the velocity will quickly approach its maximum value. It may be expected that F will go to zero and the coast period will commence very near the point at which Mode I acceleration goes to zero. For a satisfactorily compensated system it seems reasonable then to assume that Mode I will terminate very near to maximum velocity. Eqs. 16, 17, and 18 will give the time at which maximum velocity occurs, its magnitude, and the position at that instant. By setting Eq. 18 equal to zero the following relations are obtained:

$$\tan\omega_n\sqrt{K-\zeta^2}\,t' = \frac{\sqrt{K-\zeta^2}}{\zeta} \tag{19}$$

$$\sin\omega_n\sqrt{K-\zeta^2}\,t' = \frac{\sqrt{K-\zeta^2}}{\sqrt{K}} \tag{19a}$$

$$\cos\omega_n\sqrt{K-\zeta^2}\,t' = \frac{\zeta}{\sqrt{K}} \tag{19b}$$

From Eqs. 16 and 17 and the above:

$$\dot{\Theta}(t') = \dot{\Theta}_{max} = \sqrt{K}\,\omega_n\,\epsilon^{-\zeta\omega_n t'} \tag{19c}$$

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$$\theta(t') = R\left(1 - \frac{2\zeta}{\sqrt{K}}\, \epsilon^{-\zeta\omega_n t'}\right) \tag{19d}$$

or

$$E(t') = R\,\frac{2\zeta}{\sqrt{K}}\, \epsilon^{-\zeta\omega_n t'} \tag{19e}$$

where t' is the time at maximum velocity and could be found from Eq. 19 if desired. Its value will not be required.The quanities defined by Eqs. 19c and 19e may be substituted in Eq. 11 to obtain the inequality of Eq. 20. From 19c and 19e and since t' is assumed to be equal to t:

$$\left|\frac{\dot{\theta}(t')}{E(t')}\right| \simeq \left|\frac{\dot{\theta}_{01}}{E_{01}}\right| = \frac{\sqrt{K}\,\omega_n \epsilon^{-\zeta\omega_n t'}}{\frac{2\zeta}{\sqrt{K}}\,\epsilon^{-\zeta\omega_n t'}} = \frac{K\omega_n}{2\zeta}$$

Then from Eq. 11

$$\frac{K\omega_n}{2\zeta} \geq 2\zeta\omega_n \quad \text{or} \quad K \geq 4\zeta^2 \tag{20}$$

The limitation above certainly poses no problems as stated. In the limit however a criteria of $\left|\frac{\dot{\theta}_{01}}{E_{01}}\right| \simeq 2\zeta\omega_n$ would be more realistic because if $\frac{\dot{\theta}_{01}}{E_{01}} \gg 2\zeta\omega_n$ the system has driven to such a high velocity in Mode I that a large amount of overshoot is probable. It appears then that the above condition would be approached only for high values of damping ratio. This suggests that for a high ζ and a low value of K, τ_A must be larger than if a higher value of K were used. The minimum value of K is unity which poses the distinct possibility that for damping ratios less than 0.5 it would be impossible to control the amount of overshoot at this limit.

The second limiting condition from Eq. 14 is that

$$\frac{1}{\tau_A} = 2\zeta\omega_n$$

which by using Eq. 4a leads to the system transfer function of:

$$G_c = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n K s + K\omega_n^2)} \tag{21}$$

In this case the effective damping ratio of the system has been increased

[illegible]

by $\sqrt{K}$. By using the same technique as above an expression comparable to that of Eq. 20 may be reached for this condition. It is found that for either real or complex conjugate roots (from Eq. 21) it is only necessary to replace ζ by $K\zeta$ in Eq. 20 to obtain:

$$1 \geq 4\zeta^2 \tag{22}$$

This relation is considerably more significant than that of Eq. 20. For damping ratios greater than 0.5 the conditions of Eq. 22 are not met and the system may be expected to give a critical or overdamped type of response. For $\zeta > 0.5$ compensation using the nonlinear network should be satisfactory by using a cancellation lead (e.i. $\frac{1}{T_A}$ equals $2\zeta\omega_n$) and any reasonable value of K. Similar reasoning to that used above leads to the conclusion that actually at this limit $4\zeta^2$ must very nearly equal unity. If the situation exists so that $4\zeta^2 >> 1$ it may be expected that $\frac{\dot{\theta}_{o1}}{E_{o1}} >> 2\zeta\omega_n$. Eq. 10 may be rearranged to obtain:

$$\epsilon^{2\zeta\omega_n t_{c2}} = \frac{1}{1 - \frac{2\zeta\omega_n}{\dot{\theta}_{o1}/E_{o1}}}$$

As $\frac{\dot{\theta}_{o1}}{E_{o1}} >> 2\zeta\omega_n$ from above $\epsilon^{2\zeta\omega_n t_{c2}} \longrightarrow 1$

From Eq. 9b it is observed that $\dot{\theta}(t_{c2}) \simeq \dot{\theta}_{o1}$ or simply that Mode II operation was nearly at constant velocity. This is the extreme of course, but shows quite clearly that for damping ratios much less than 0.5 a large amount of overshoot might be expected. Eq. 22 may be written more properly for this limiting condition as:

$$1 \simeq 4\zeta^2 \tag{23}$$

Since small values of damping ratios were ruled out for the first limiting condition investigated, it follows that the nonlinear network will not be very effective for compensating those systems having damping ratios much less than 0.5. It was shown that for $\zeta \simeq$ 0.5 proper compensation

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will be obtained with an open loop zero at approximately $2\zeta\omega_n$ and the response will be relatively independent of the location of the new pole although a higher value of K may be expected to give a slightly faster response. As the magnitude of ζ is increased the open loop zero must be moved toward the origin of the complex plane to obtain the desired response.

Experimental results verified the above conclusions. The design curves of Section 6 clearly demonstrate the effect of varying the damping ratio. Appendix I contains a tabular summary of the pole and zero locations for several compensated systems.

3.5 Discussion

Although a general analysis has not disclosed any practical method for design, some ideas as to performance and regions of applicability have been obtained. It has been shown that within a region where compensation is possible the system response will be relatively independent of the magnitude of the input. Also it was shown that the nonlinear network can be expected to be effective only when $\zeta > 0.5$. This is of course only an approximation, but the most important assumption was that cutoff, t_{c_1}, occurred at maximum velocity. The actual cutoff was found to occur at a slightly lower velocity with a somewhat smaller error at that point which tends to extend the range to a slightly lower damping ratio. Eq. 22 is independent of K which indicates that for all values of K this limitation holds. It is reasonable to assume that through the range of practical values for K that the system response will be virtually unchanged for the same maximum overshoot. From Eq. 19 it is observed that the time to maximum velocity will be quite small compared to the time in Mode II (Eq. 10a). More is mentioned on this subject in the following section.

It is possible to surmise that over the region of applicability, that, in general, as γ_A is increased with the damping ratio held constant, the

rise time during Mode I will be decreased and at some point will result in excessive overshoot. Also, as the damping ratio is decreased the value of τ_A may be expected to decrease for a given desired response. The limit will be when the introduced zero cancels the open loop pole at $2\zeta\omega_n$. Appendix II contains a sample analysis illustrating the validity of the relations derived in this section.

4.0 PHASE PLANE ANALYSIS

During the evaluation of the nonlinear network the phase plane provided a pictorial representation of how the system operated. It was invaluable as a qualitative tool although of relatively little value quantitatively due to the third order characteristic equations for all except Mode II operation.

Eq. 9, the basic defining equation in Mode II, indicates the possible pictorial value of the phase plane. It may be manipulated to obtain:

$$\frac{\ddot{\theta}_c}{\dot{\theta}_c} = \frac{d\dot{\theta}_c}{d\theta_c} = -2\delta\omega_n = N \tag{24}$$

On the phase plane (θ_c vs $\dot{\theta}_c$ or E vs $\dot{E}$ for a step input) this simple relation shows that Mode II will be a straight line with a slope of $-2\delta\omega_n$. The problem would be very simple if the initial conditions for Mode II were known. However, they are determined by the system characteristics in Mode I which are described by third order equations. Eq. 7 gives the answer, qualitatively at least, to this problem. As discussed in Sect. 3.1 the time to cutoff will be a constant for a given system regardless of the magnitude of the input. Consider then a three dimensional plot of position, velocity, and acceleration for various magnitudes of inputs. The cutoff points must fall in a plane; therefore, a projection of position and velocity to a two space plot will not effect the above relations. It is perhaps more straightforward to think in terms of error since a step input was considered. For an initial error of magnitude R a projected response as shown in Fig. 4-1 might be expected for a compensated system for the first two modes of operation. From Fig. 4-1 it is easy to visualize the basic requirements for a satisfactory system. It shows that for a critically damped response the values of error velocity and error at t_{c1} must be such that their quotient is equal to $-2\delta\omega_n$. Likewise, from Fig. 4-1, it

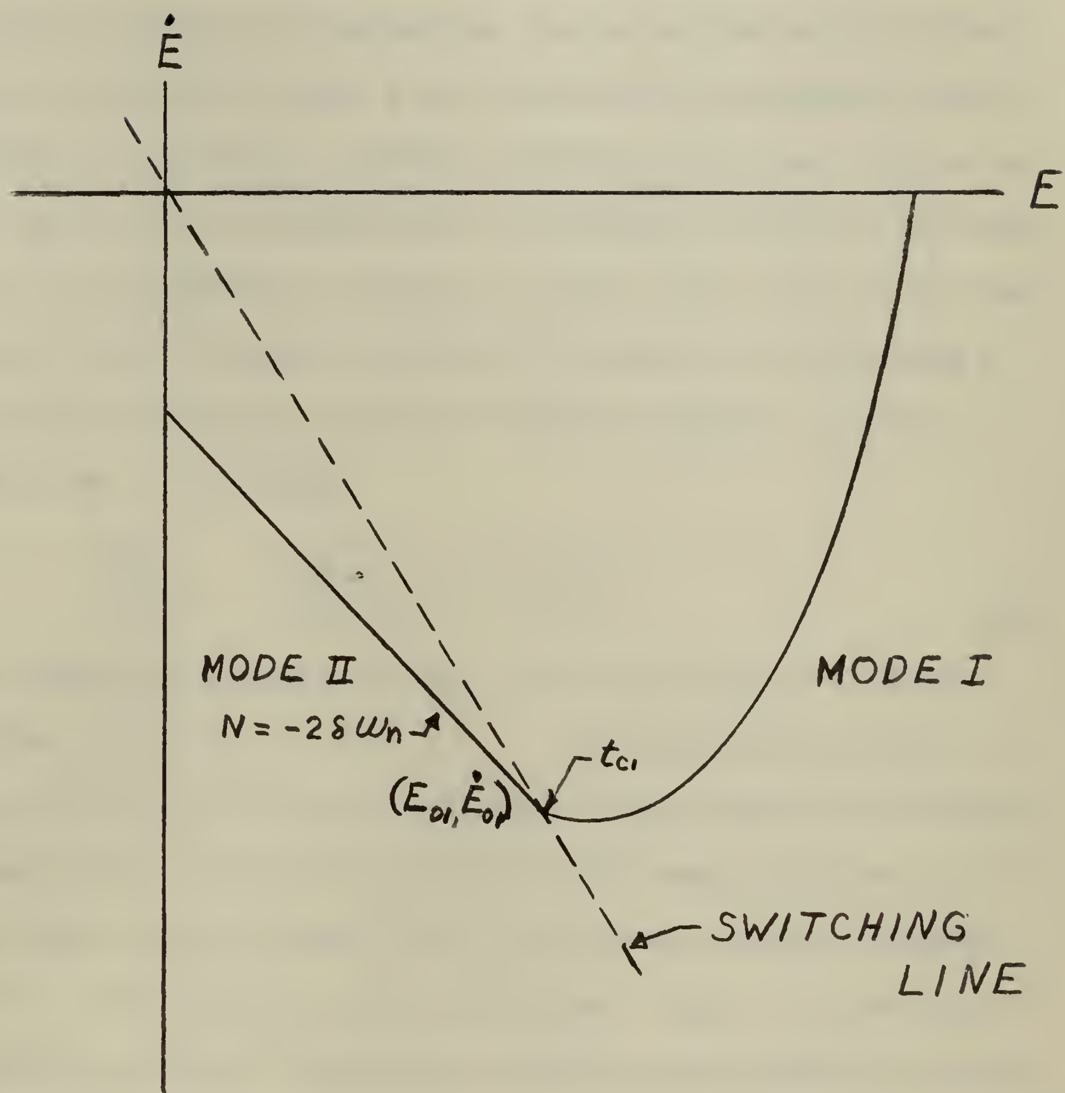

Fig. 4-1 Phaseplane plot for Mode I and Mode II

is observed that if $\left|\dot{E}_{o1}/E_{o1}\right|$ is less than $2\,\xi\,\omega_n$ the system will most likely act in a manner somewhat analogous to a chattering relay. That is, it will try to coast in Mode II but the error will be decreasing slower than the capacitor is discharging, thus a positive signal is again transmitted. The resultant Mode I type operation will immediately drive it once again to a coast type operation. Theoretically there could be no overshoot and the time of response would be infinite for both of the above conditions. The mathematical relations for Mode II (Eq. 10a) verify this. Since the most rapid response is desirable it would not be reasonable to compensate for an overdamped response. Thus the identical conclusion to that given by Eq. 11 is reached:

$$\frac{\dot{\theta}_{o1}}{E_{o1}} = \left|\frac{\dot{E}_{o1}}{E_{o1}}\right| \geq 2\,\xi\,\omega_n \qquad (11)$$

The other possibility suggested by Fig. 4-1 is the condition as drawn, that is, that $\left|\dot{E}_{o1}/E_{o1}\right| > 2\,\xi\,\omega_n$. For this situation Mode III must exist and there will be some overshoot. If it is desired to maintain the overshoot within reasonable proportions (for example 5% as was done in the design work) it follows that $\left|\dot{E}_{o1}/E\right|$ must be only slightly greater than $2\,\xi\,\omega_n$ or the overshoot will be excessive. Fig. 4-5 illustrates a system where the nonlinear compensation resulted in approximately 20% overshoot. The conclusion here then is identical to that reached in Section 3.4, that $\dot{E}_{o1}/E_{o1} \overset{\text{slightly}}{\geq} 2\,\xi\,\omega_n$. To extend the discussion to Mode III presents more difficulty and since the mathematics have been well defined it seems sufficient to state that Mode III will be similar to Mode I operation. It appears that the larger the value of K, the greater the braking force that might be expected in Mode III. Fig. 4-2 pictorily summarizes the entire operation for a second order system. A hypothetical

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system is assumed and qualitative operation is sketched for inputs of two different magnitudes.

The phase plane also gives an indication of the response time that might reasonably be expected for a given system. From the preceding figures it is observed that the time, t_{c1}, will normally be small compared to the time for Mode II operation, t_{c2}. Assume then that t_{c1} is negligible and the response is as shown in Fig. 4-3. The approximate time may then be determined:

$$\dot{E} = \frac{dE}{dt} \quad or \quad \dot{E}_{ave} \simeq \frac{\Delta E}{\Delta t}$$

$$\Delta t = t_s \simeq \frac{\Delta E}{\dot{E}_{ave}} \quad and \quad t_s \approx \frac{1}{\zeta \omega_n}$$

Or more appropriately to consider non-dimensionalized time:

$$\omega_n t_s = \frac{1}{\zeta} \tag{25}$$

Examination of the design curves in Section 6 verify this to be a reasonably accurate approximation.

Examples of phase plane plots for several systems are shown in Figs. 4-4, 4-5, and 4-6. Information for all of the curves was obtained from analog computer solutions (Section 5). The term, "Region of Applicability", is used to define the region, in terms of damping ratio, where the basic nonlinear circuit provided satisfactory compensation (overshoot limited to 5%), and the response time was less than the optimum linear lead for the specified response. Fig. 4-4 shows the effectiveness of the nonlinear compensator in "speeding up" a sluggish system. Fig. 4-5 is representative of the curves obtained for systems outside the above defined "Region of Applicability". (The oscillograms of Fig. 8-2 show a typical transient response for this type of system.) The proposed compensator actually improved the response, but the overshoot was considerable. In comparison

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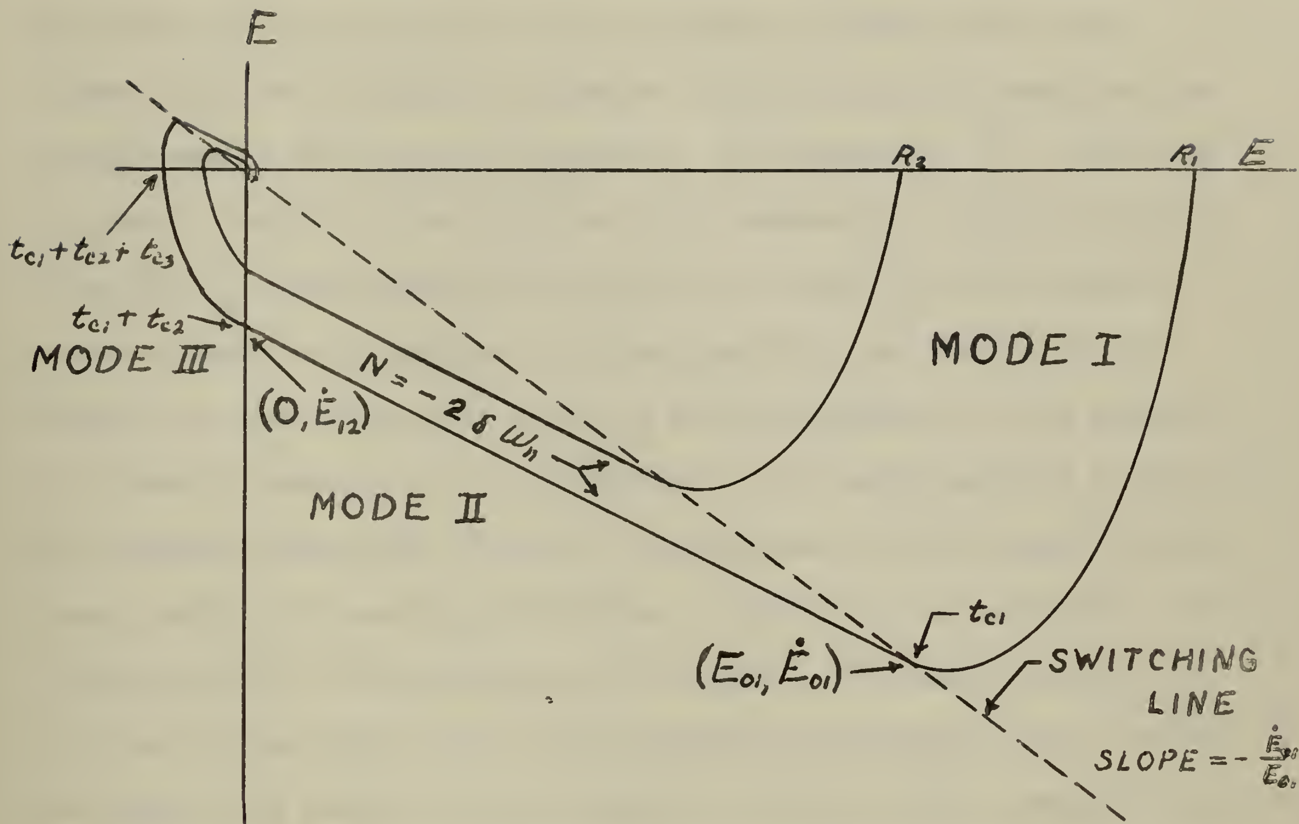

Fig. 4-2 Phaseplane sketch of nonlinear network operation for a second order servo system

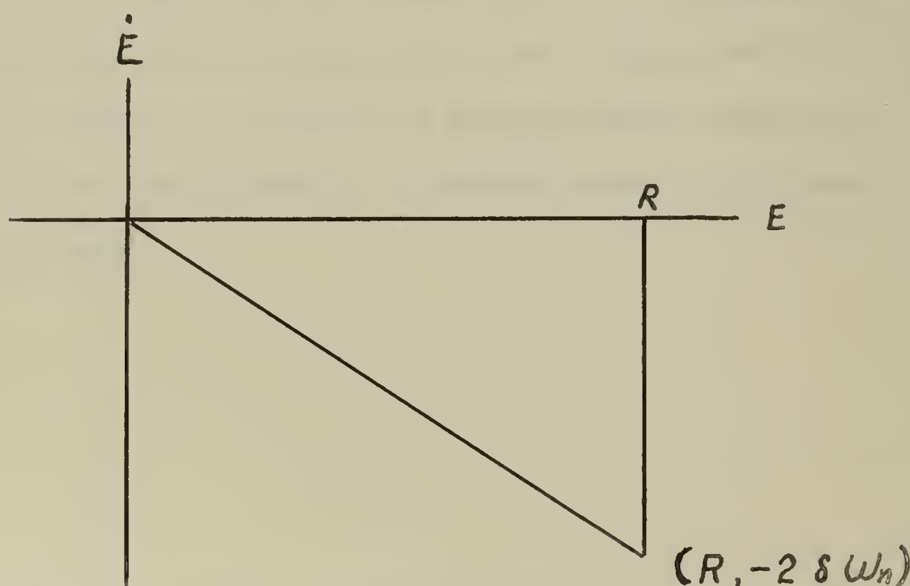

Fig. 4-3 Phaseplane sketch to determine the approximate response time

Fig. 4-5 also shows a linear lead that provided excellent compensation. The nonlinear system appears to have actually followed close to the "coast line" even though the parameters indicate that the capacitors discharged faster than the error decreased. An increase of r_A would lead to greater overshoot since the coast would commence at a higher velocity. A smaller r_A would decrease the velocity at cutoff, but the position would be nearer correspondence and there would be less braking during Mode III so that performance could not be much improved. If the capacitor is allowed to become zero the performance is of course identical to that of the original uncompensated system. The nonlinear and the linear networks used to obtain the oscillograms of Fig. 8-2 had identical parameters. The coast period for the nonlinear is well defined. The maximum velocity is as great for the linear lead, but the capacitor discharge occurs to brake the system more rapidly than was possible during the coast operation for the nonlinear. A slight adjustment of parameters for the linear network would give an improved response over that shown.

Fig. 4-6 gives visual proof that the cutoff time, t_{c1}, is virtually constant as shown mathematically and that position and velocity at the termination of Mode I is directly proportional to the magnitude of the step input, R. Also, note that the percentage of overshoot shows little change over a relatively wide range.

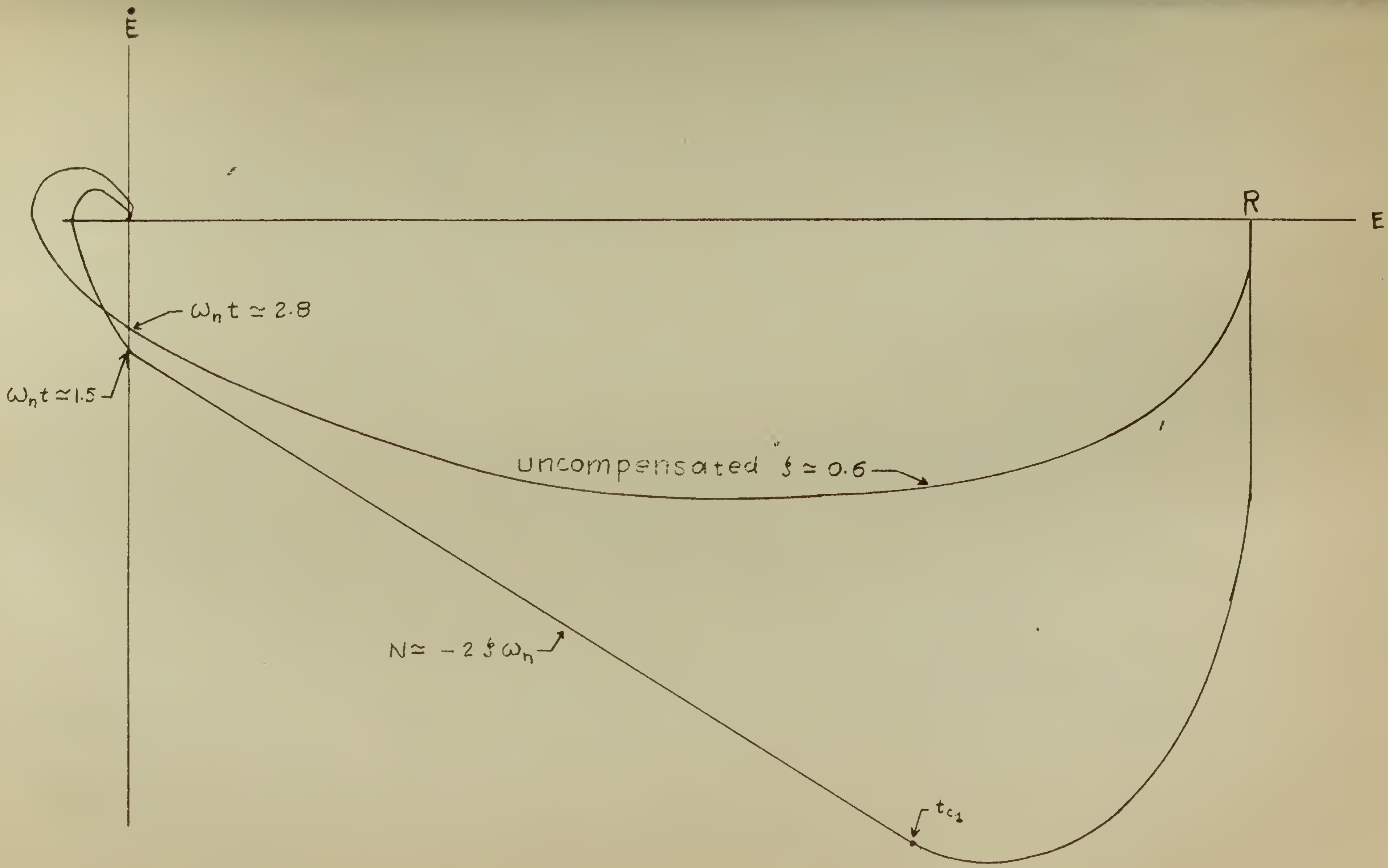

FIG. 4-4

Effect of Non-Linear Compensation Within "Region of Applicability"

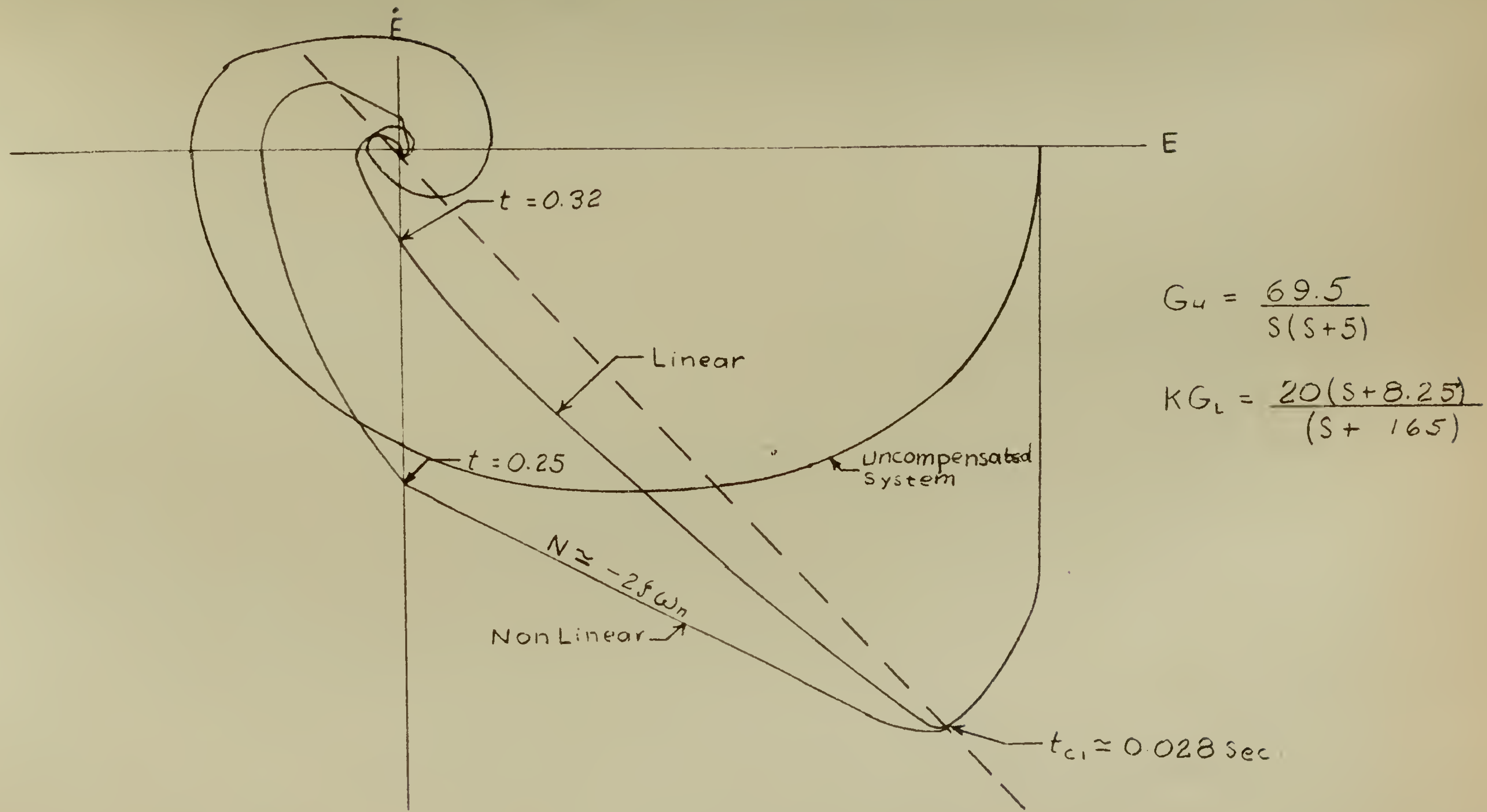

FIG. 4-5

Comparison of Linear and Non-Linear Lead Compensation Outside Region of Applicability of Non-Linear Network

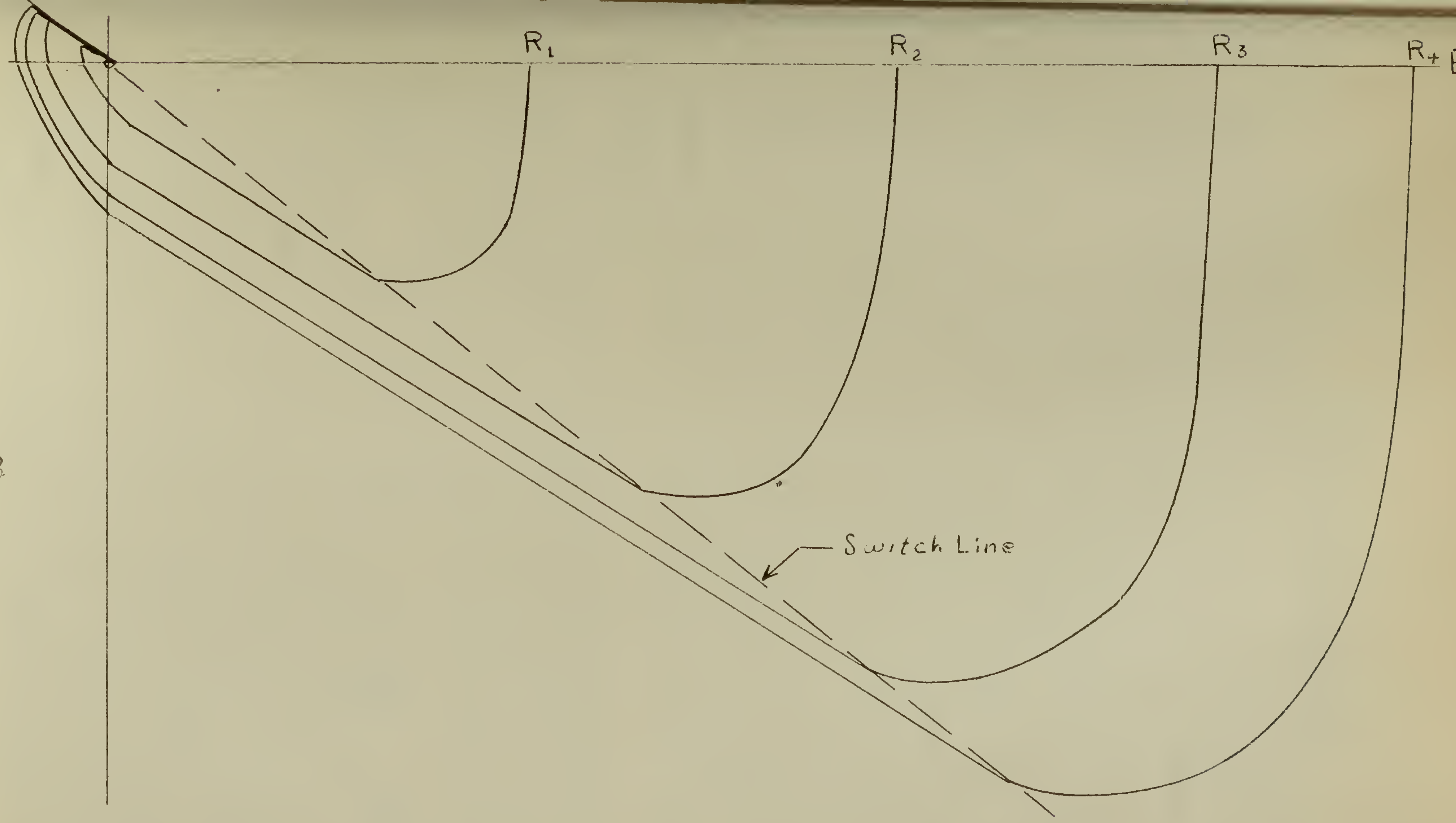

FIG 4-6

Effect of Varying Magnitude of a Step Input to a Non-Linear Compensated Circuit ($\zeta \simeq 0.6$)

5.0 ANALOG COMPUTER EVALUATION

Even though a mathematical analysis is possible for a simple servo system, this method would never be used for compensation purposes. The analog computer provides a realistic approach to studying the system. All transfer functions can be simulated on the computer, but in order to study the effects of changing the compensation parameters the actual components were used in the lead networks. The diodes were a 6AL5 Twin Diode tube. A diagram of the analog computer set up is shown in Fig. 5-1.

The use of actual components in part of the analog computer did introduce some problems. In order to study the transient response it was necessary to scale computer time by a factor of ten. For the lead network parameters to be on the same time scale, required only that the value of C_A be ten times larger than the C_A for real time. This permitted the use of components that were in the range of readily available laboratory resistors and capacitors.

Since the study was based on the diodes being the only nonlinear element in the system, all quantities had to be scaled to insure that no amplifier saturated at any time. This was insured by using a 10 volt step input with scale factors as follows:

$$\alpha_t = 10 \qquad \alpha_\theta = \alpha_E = \alpha_F = \frac{1}{40} \qquad \alpha_{E_1} = \frac{1}{20} \qquad \alpha_{\dot{\theta}} = \frac{1}{4}$$

E, F, and E_1 refer to quantities as labeled on Fig. 5-1.

The problem of loading, while not difficult, was important because of its effect on the overall accuracy of the results. For example, it is assumed that the input resistance to amplifier number two is combined with the output resistor of the nonlinear network to give the actual R_o. In order to decrease the effect of the input resistor to the next stage, an amplifier with an input resistor of ten megohms and gain of unity was

placed after the lead network.

For the first runs on the computer the value of lead network attenuation, α, was chosen to be one tenth and was held constant. Once the value of α had been chosen a value of R_A was arbitrarily picked, thus fixing the value of R_o, since

$$\alpha = \frac{\dfrac{R_o R_3}{R_o + R_3}}{\dfrac{R_o R_3}{R_o + R_3} + R_A} \tag{26}$$

The value chosen for R_A was five megohms. Substituting $\alpha = \frac{1}{10}$; $R_A = 5$, $R_3 = 10$; R_o is found to be .513 megohms. It is necessary to compute the attenuation exactly even though $R_3 \gg R_o$ since K must equal $\frac{1}{\alpha}$ if the velocity lag error after compensation is to remain unchanged. The value of five megohms for R_A is not critical; any convenient value can be picked to substitute into Eq. (26) to solve for R_o. Compensation depends only on the time constants, τ_A and τ_B. With α fixed these time constants are varied by adjusting only C_A.

Next a second order servo was simulated on the computer and compensated by varying C_A to give a response with a maximum overshoot of 5%. This procedure was repeated for second order systems with damping ratios ranging from 0.1 to 1.0 with the following data recorded for each run:

ζ	damping ratio of combined motor-load combination
ω_n	natural frequency of combined motor-load combination
t_s	settling time (time to reach 95% of final value)
t_L	settling time for linear lead compensation
C_A	capacitance of compensation network

By using a normalizing procedure to be explained later (See Section 6) it was possible to present the data in such a manner that design parameters for any second order system can be chosen by knowing the damping ratio

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$$[illegible] \quad (1)$$

$$[illegible] \quad (2)$$

$$[illegible] \quad (3)$$

$$[illegible] \quad (4)$$

[illegible]

and natural frequency of the uncompensated system.

For this nonlinear lead network to be of value it must be better than any less complicated network. With this thought in mind the same data was collected for a linear lead network.

The nonlinear compensation was found to give a shorter settling time than a linear lead for damping ratios greater than 0.5. The linear lead was superior for those less than 0.45. This can be seen from the design curves, Figs. 6-3, 6-4, and 6-5. In the region of $0.45 < \zeta < 0.5$ the two networks gave approximately equal settling times. The previously defined "Region of Applicability" was thus determined.

Even though the computer technique was by trail and error it was not difficult to find the value of C_A to give the desired response. The technique was as follows. The second order damping ratio and natural frequency were chosen for a particular run and potentiometers 4, 5, and 6 were set accordingly. The magnitude of the input signal was 10 volts. Potentiometer 3 was set by taking the output of amplifier 1 through R_A across R_o and into amplifier 2, and adjusting potentiometer 3 to make E_1 equal to 5 volts. (Since $\alpha_\theta = \frac{1}{40}$ and $\alpha_{E_1} = \frac{1}{20}$). This eliminated any error due to inaccuracy in measuring the 10 megohm resistors in amplifier 2, thus insuring that K equaled exactly $\frac{1}{\alpha}$. Next the loop was closed and the second order response was run and checked against the second order curves [8]. This checked the computer setup to insure that the desired second order system was actually set on the computer. Then the nonlinear network was put in the system and C_A varied to give the desired response.

For large damping ratios C_A was relatively large. This was as anticipated based on the previous mathematical analysis. A large damping ratio means a system with considerable inherent damping; therefore, it can be

driven for a longer period before it cuts off and coasts. If $R_A \gg R_O$ the driving time is proportional to $R_O C_A = \tau_B$. As the damping ratio decreases with $2\zeta\omega_n$ held constant C_A must decrease so the driving time will be reduced in order to limit the overshoot to 5%. There is a lower limit of ζ below which the overshoot cannot be decreased by further decreasing the value of C_A. This can best be visualized by observing that as $C_A \rightarrow 0$ the original second order system is approached, since with $C_A = 0$ we only have the second order system with the signal attenuated by the value of α and amplified by $K(= \frac{1}{\alpha})$. It was found that the overshoot could not be held to 5% for damping ratios less that 0.45 which confirms the conclusion reached mathematically in Sect. 3.4.

Similar data was recorded for $\alpha = \frac{1}{5}$ and $\alpha = \frac{1}{20}$. While the response time for a given damping ratio does vary slightly with the value of α the damping ratio below which the second order system cannot be compensated with the nonlinear network does not vary appreciably.

In addition to studying the transient response to obtain compensation parameters and settling time, the analog computer was used to obtain data for a study of the system using the phase plane. This was accomplished by simultaneously recording θ_c and $\dot{E}$ and plotting the points on the phase plane.

The following section will describe three design charts (one for each α) based on the data collected using the computer technique explained above.

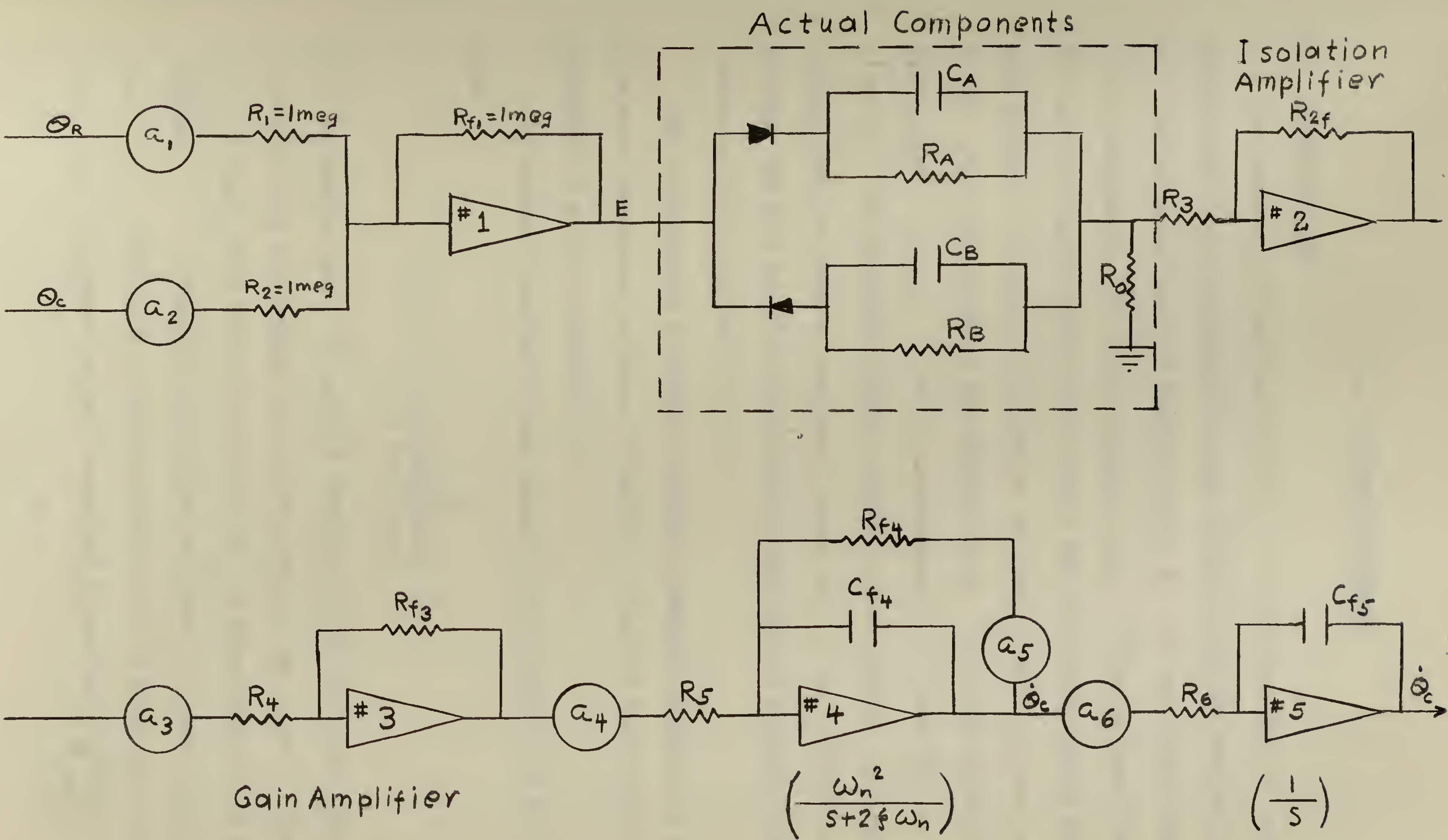


Fig. 5-1 Analog simulation of second order servo with nonlinear lead compensation

6.0 DESIGN PARAMETERS

6.1 Design Curves

The problem of displaying the data in a manner so that it could be quickly and easily interpreted to give design parameters was next considered. The variables are: (1) damping ratio, ζ , of the second order system to be compensated, (2) natural frequency, ω_n , of second order system, (3) settling time of compensated system, and (4) capacitance of nonlinear lead network. For each set of data α was held constant, K was made equal to $\frac{1}{\alpha}$, and the overshoot was limited to 5%. The value of the capacitor can be made a function of time by multiplying by R_A since $R_A C_A$ is defined to be τ_A , the discharging time constant of the network. By non-dimensionalizing time and using a coordinate system with ζ as ordinate and $\omega_n t$ as abscissa the data for all second order systems can be displayed using two curves as illustrated in Fig. 6-1.

The corresponding normalized transient response obtained by using design curve parameters is shown in Fig. 6-2. Proof of this normalizing technique is given in Section 6.2.

The transfer function for any second order system can be put in the form:

$$G_{ML} = \frac{\omega_n^2}{S(S + 2\zeta\omega_n)}$$

With the damping ratio and natural frequency known, the curve (Fig. 6-1) is entered with the damping ratio. The abscissa for point (1) gives the value of $\omega_n \tau_A$ from which the value of τ_A is found directly to be $\frac{(\omega_n \tau_A)}{\omega_n}$. Likewise the abscissa for point (2) gives $\omega_n t_s$ and settling time is calculated. Once the value of τ_A is known the circuit parameters can be determined quickly since $\tau_A = R_A C_A$ and $\alpha = \frac{R_o}{R_A + R_o}$. A value

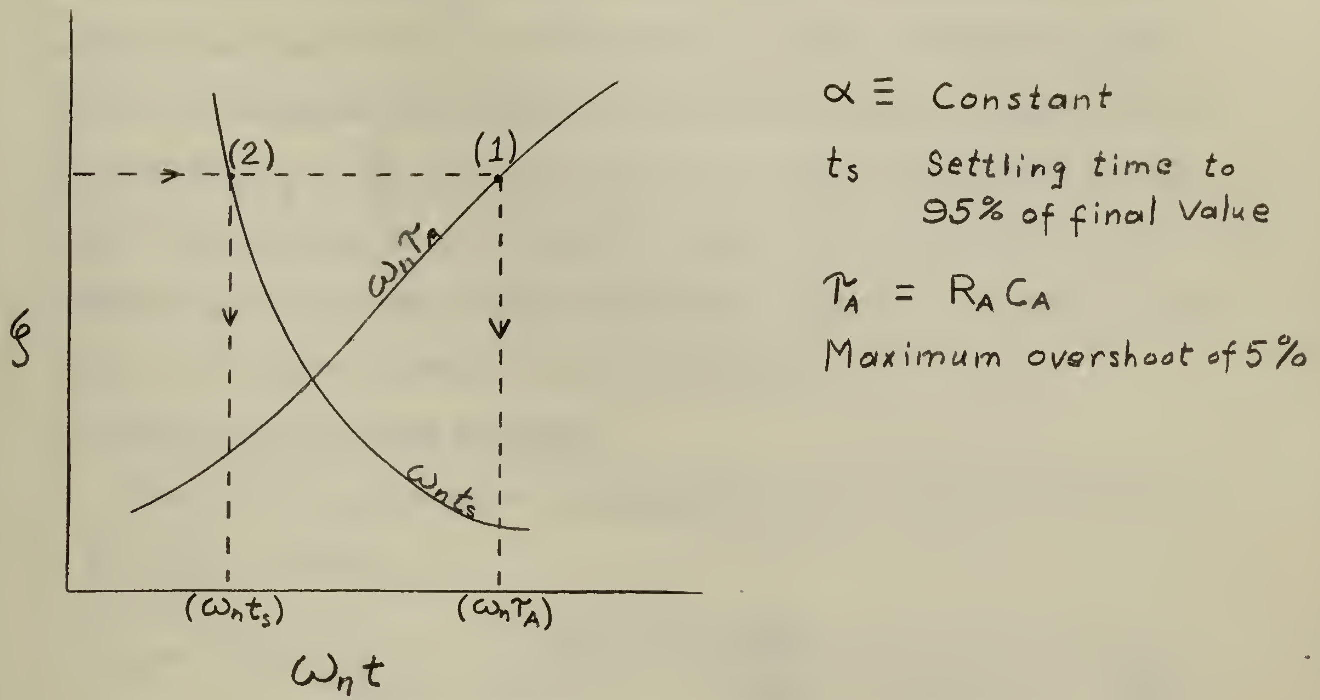


Fig. 6-1 Sketch of design curves for second order servo system

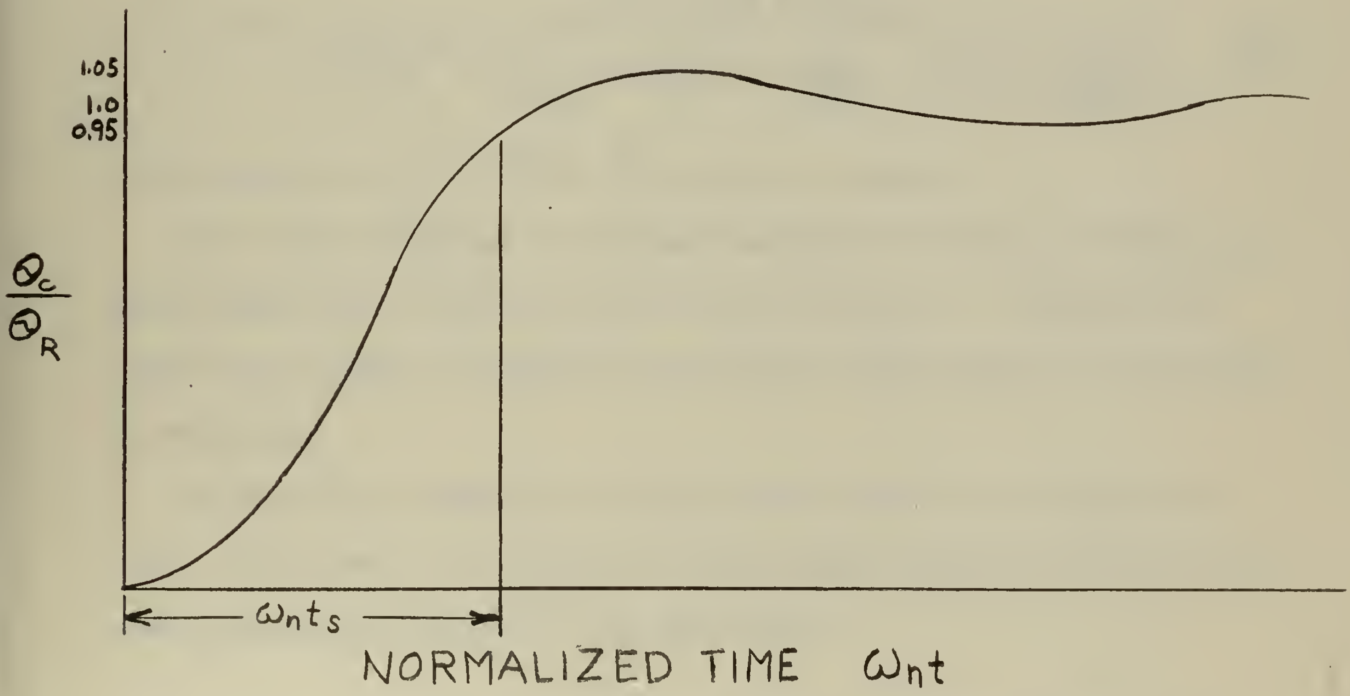


Fig. 6-2 Transient response obtained using design curve parameters

may be chosen for either R_o or R_A and the other is immediately fixed to satisfy the value of α for the particular curve. Values of R_A from five to ten megaohms were found convenient for analog simulation. Once R_A is fixed, $C_A = \frac{\tau_A}{R_A}$ and all parameters are known. By use of the design curves of Figs. 6-3, 6-4, and 6-5 the parameters required to compensate any second order system can be determined quickly. The settling time for a maximum overshoot of 5% is also known.

6.2 <u>Proof of Normalizing Procedure</u>.

The normalizing procedure explained above may be proved as follows: [10, 12] . From Eq. 3

$$G_c = \frac{K\omega_n^2\left(s + \frac{1}{\tau_A}\right)}{s^3 + \left(\frac{1}{\tau_B} + 2\zeta\omega_n\right)s^2 + \left(\frac{2\zeta\omega_n}{\tau_B} + K\omega_n^2\right)s + \frac{K\omega_n^2}{\tau_A}}$$

Let $s = p\,\omega_n$. Then: $K = \frac{1}{\alpha}$ $\tau_B = \frac{\tau_A}{K}$

$$G_c = \frac{K\omega_n^3\left(p + \frac{1}{\omega_n\tau_A}\right)}{\omega_n^3\left[p^3 + \left(\frac{K}{\omega_n\tau_A} + 2\zeta\right)p^2 + \left(\frac{2\zeta}{\omega_n\tau_A} + 1\right)Kp + \frac{K}{\omega_n\tau_A}\right]} \tag{27}$$

which shows that: $G_c = f(\omega_n\tau_A, \zeta)$ since K is fixed.

For a given response (a maximum overshoot of 5% in our case) and a given damping ratio there is a fixed value of $\omega_n\tau_A$. Similar proof holds for all modes of operation, therefore, design data can be displayed as described.

As previously stated the nonlinear lead network was compared to a linear lead network. The curves for the linear network are drawn on the same graph and may be used in the same manner.

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

6.3 Sample Problem Using Design Curves

The design curves define the parameters and the settling time for a transient response of the general shape of Fig. 6-2 for a unit step input signal. The settling time is defined as t_s for the nonlinear network and t_L for the linear lead network. Likewise the time constants τ_A and τ_L refer to the nonlinear and linear lead networks respectively. A simple example is chosen to demonstrate the use of the curves.

Assume: $G_u = \dfrac{10^2}{s(s+12)}$

Then: $\omega_n = 10$ $\qquad 2\zeta\omega_n = 12$ $\qquad \zeta = 0.6$

For the fastest response α will be chosen as 0.05. Enter Fig. 6-5 with the damping ratio of 0.6 to obtain $\omega_n \tau_A = 1.58$ and $\omega_n t_s = 1.47$. Immediately it follows that $\tau_A = .158$ sec and $t_s = .147$ sec. Let $R_A = 8$ megohms (or any reasonable value), $C_A = \dfrac{\tau_A}{R_A} = 0.02\,\mu f$. Since $\alpha = 0.05 = \dfrac{R_o}{R_A + R_o}$, R_o is computed to be 0.421 megohms. An additional gain of 20 must be supplied to maintain the desired steady state gain. The resulting compensated system may be expected to have a settling time of 0.147 seconds (to 95% of final value) compared to 0.25 seconds for the original system.

If greater overshoot may be tolerated the value for τ_A should be slightly greater; if less overshoot is desired τ_A must be decreased. The response time will be decreased for the former and increased for the latter.

[illegible]

[illegible]

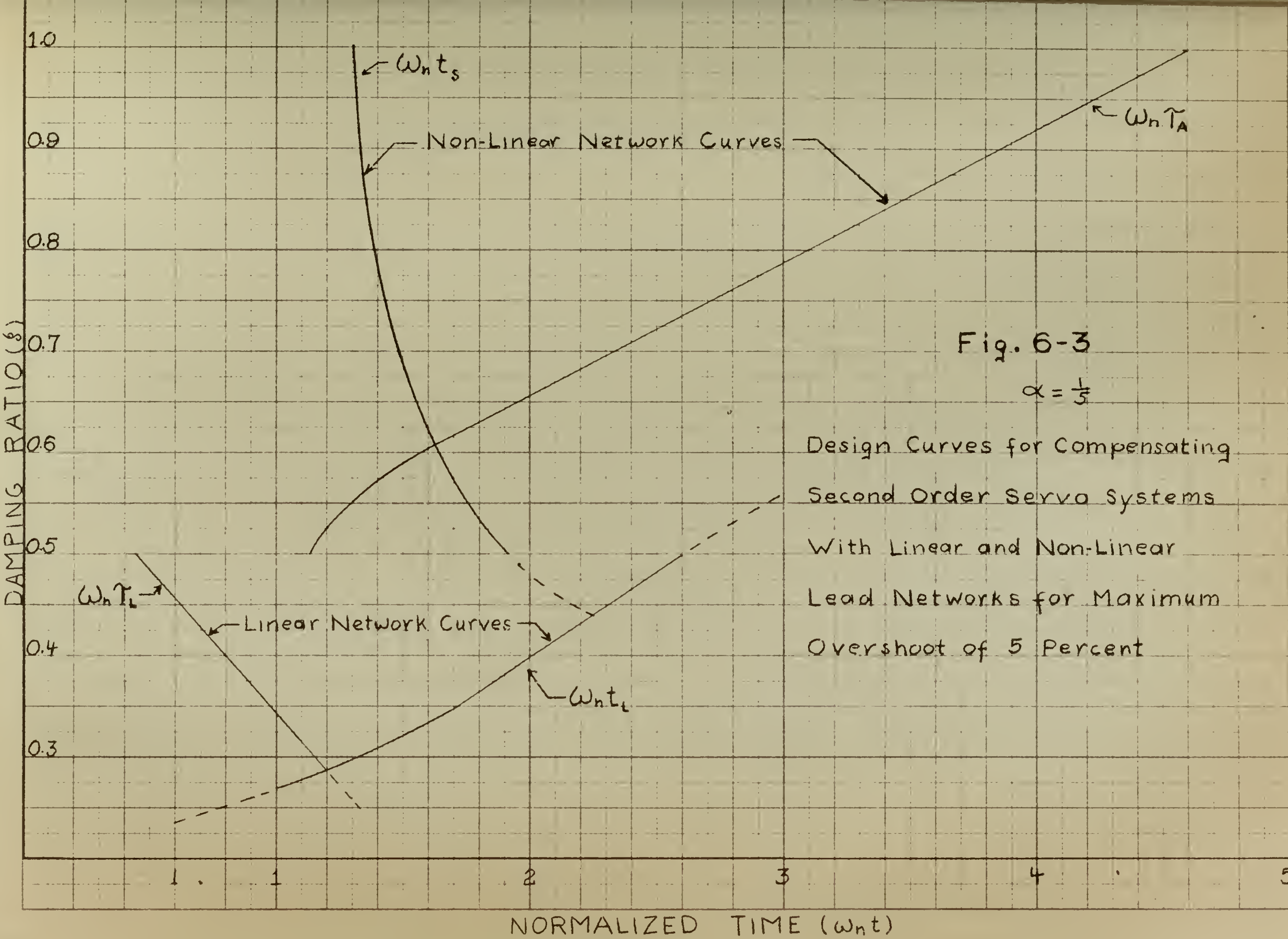


Fig. 6-3

$\alpha = \frac{1}{5}$

Design Curves for Compensating Second Order Servo Systems With Linear and Non-Linear Lead Networks for Maximum Overshoot of 5 Percent

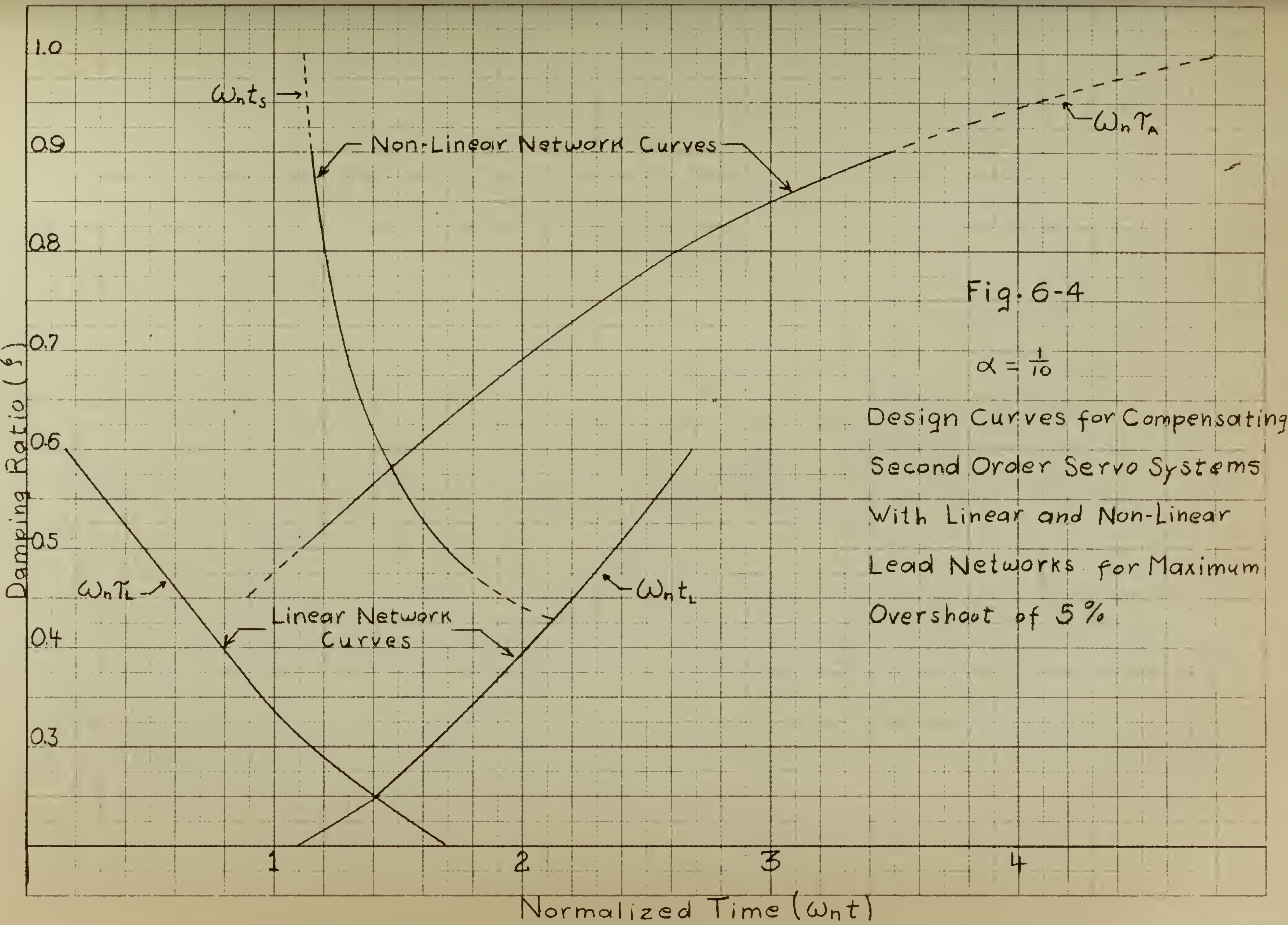


Fig. 6-4

$\alpha = \frac{1}{10}$

Design Curves for Compensating Second Order Servo Systems With Linear and Non-Linear Lead Networks for Maximum Overshoot of 5%

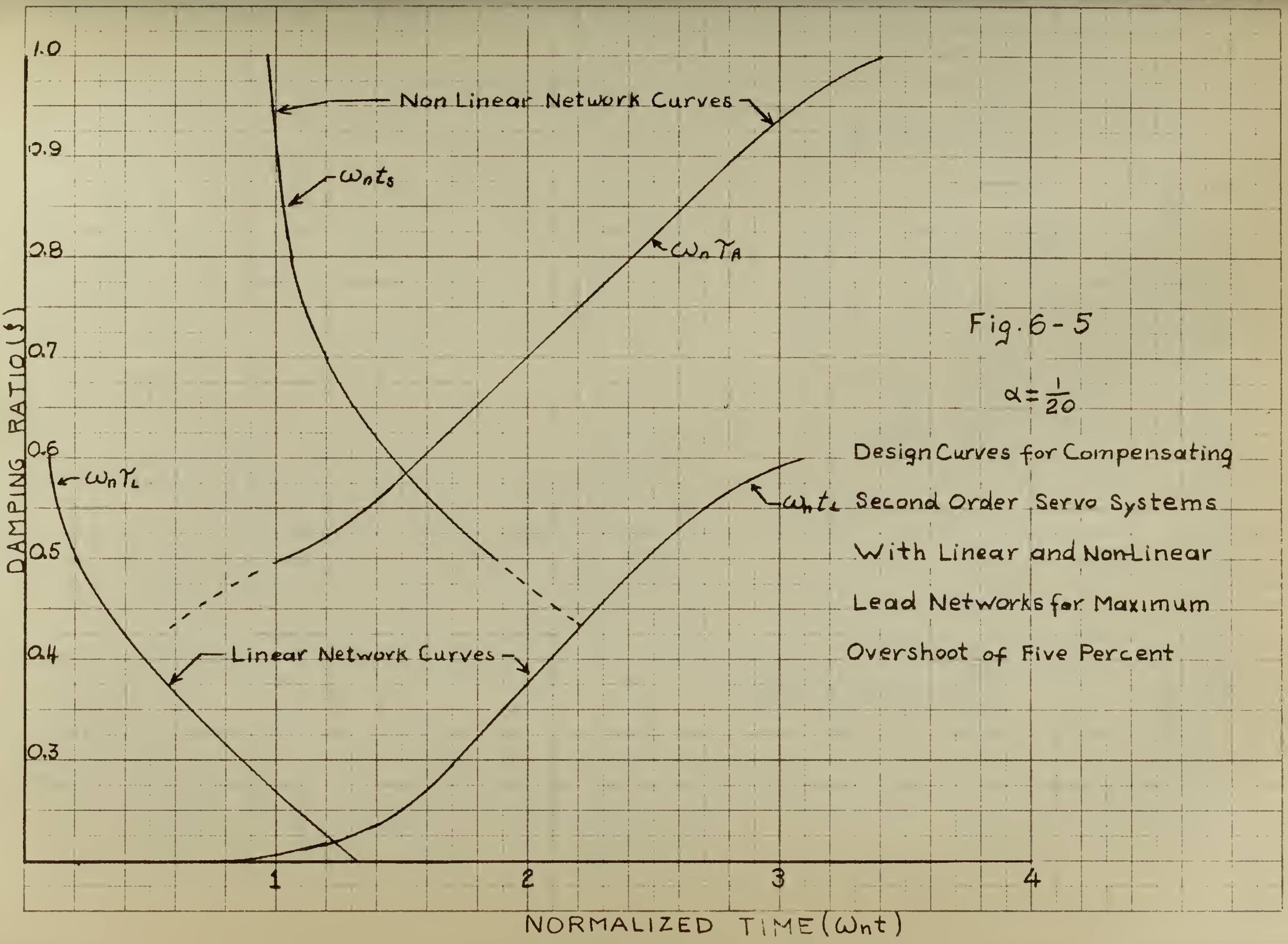


Fig. 6-5

$\alpha = \frac{1}{20}$

Design Curves for Compensating Second Order Servo Systems With Linear and Non-Linear Lead Networks for Maximum Overshoot of Five Percent

7.0 POSSIBLE MODIFICATIONS TO BASIC CIRCUIT [1]

The modification shown in Fig. 7-1 was investigated experimentally for several second and third order systems.

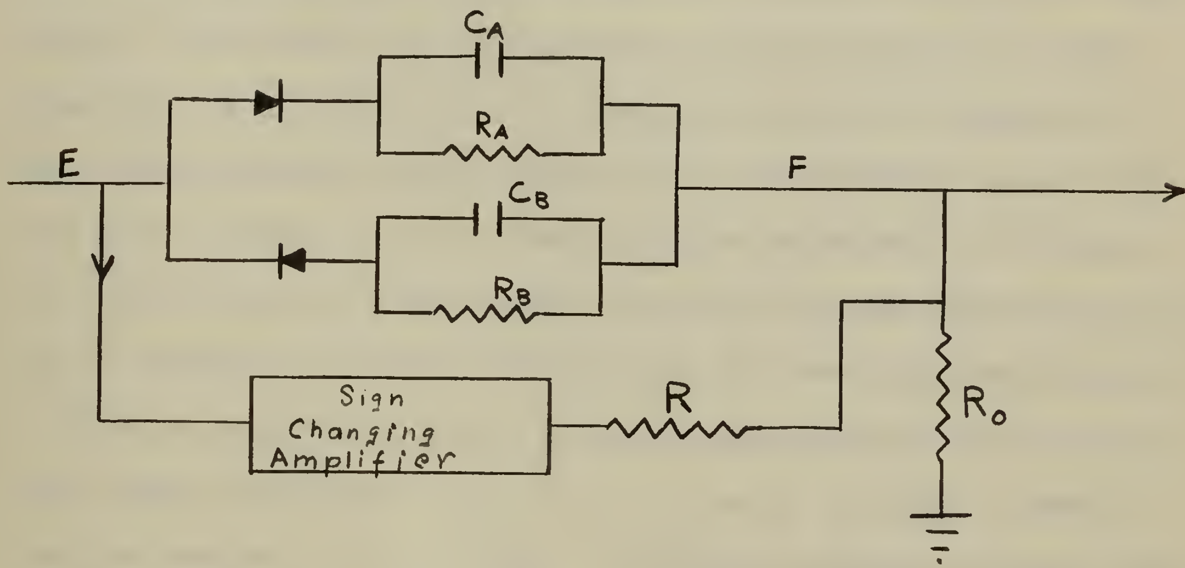
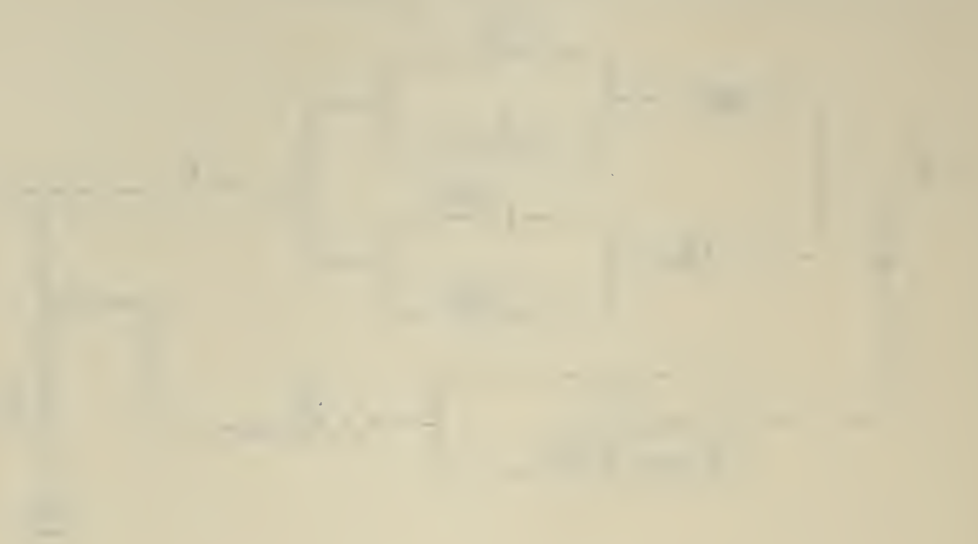


Fig. 7-1 Basic network with sign changing amplifier for added damping.

The function of the sign changing amplifier in the feed forward loop is to provide added damping when the error becomes small. As long as either diode is conducting the added loop is relatively ineffective because the impedence of the diodes and capacitor will be much smaller than that of the resistor, R. When the diodes are open the sign changing amplifier will provide a signal at the output terminal which will have a polarity opposite to that of the error voltage. This signal will provide additional damping by reversing the output torque. The system will then decelerate more rapidly in Mode II than would be possible with only inherent damping. The feed forward loop can make the system unstable; however, this will be avoided if the capacitors discharge at about the same rate that error decreases so that the diodes may conduct to override any negative error signal transmitted by the sign changer.



Sufficient experimental data was obtained to allow some general observations on the operation of this modified circuit. The network was simulated on the analog computer and several systems successfully compensated. It was found that a satisfactory value for R was two to three times that of R_o and that the sign changer gain was most effective between unity and about three. As the gain was increased it appeared that the gain tolerance to retain stability became quite small. This seems to indicate that for a gain of much more than unity the distinct possibility of instability exists if the amplifier gain deviated from linearity or if there is significant drift. In several cases introduction of saturation caused instability. (See Fig. 8-3 Curve B) Although no detailed analysis was made of this situation intuitive reasoning provides a possible explanation for this phenomena. For heavy saturation it may be expected that the proper diode will open at about the same instant as without saturation. The output shaft position will not be appreciably different than without saturation, but the maximum velocity may be considerably lower. The sign changer will then become effective at a lower velocity than if saturation were not present. The added damping will occur too soon and the error cannot reach zero since the capacitors will be discharging too slowly to pass the required overriding positive signal. The sign changer will cause the system to drive so as to increase the error until the sign changing amplifier saturates. An attempt was made to minimize the stability problem by introduction of a limiter on the sign changing amplifier [15] . Since the effect it provides was desired only when the error becomes small its maximum output was limited to about 25% of the maximum error expected, regardless of the gain of the amplifier. This limiter appeared to be quite successful in removing the sensitivity to higher gains and in the systems

[illegible]

checked seemed to remove the instability problem where saturation was present. Very little effect was noted in the response attainable with or without the limiter.

The modified circuit was successful in improving the transient response for all damping ratios. For sluggish systems the added damping modification decreased the response time slightly compared to the basic nonlinear circuit. (Fig. 8-4). This was true regardless of the system order. It appeared that the modified circuit would not be desirable whereever the basic network would satisfactorily limit the maximum overshoot. In general, the added circuit complexity would not justify the small improvement possible in response time.

For low damping ratios the modified network may well have considerable merit for some applications. It was possible to obtain very good transient responses for low damping ratios, both for second and third order systems. Compensation with the basic network in this region was not possible within the limitations set forth previously as the maximum overshoot exceeded 5%. The responses obtained with the modified circuit were not as good as that with a linear lead but were comparable. Figs. 8-5 and 8-6 indicate the small difference between a straight lead and the modified nonlinear lead. Obviously, whenever possible the straight lead would be used, but the nonlinear circuit may well be used to advantage in those systems where a d-c lead network cannot be used. This is taken up in more detail in Sec. 8.

The parameters used for the modified circuit were close to those for the basic circuit for high damping ratios although a slight increase in $\omega_n \tau_A$ helped to improve response time. For lower damping ratios extrapolation of the design curves will in general give an excellent starting

point for trial and error optimization. Some adjustment will be required since the sign changer gain and resistor, R, are now two additional variables.

7.2 Modification to Reduce Noise

Fig. 7-2 indicates a proposed modification to reduce noise. [1, 2,] Both the basic network and modified circuit of Fig. 7-1 have the usual limitations of RC circuits as to noise sensitivity. The diodes will rectify and the capacitors will store noise superimposed on the error voltage causing the circuit to be insensitive to error voltages smaller than the noise. The modified circuit should lessen the noise problem. The bypass resistor will have relatively small effect when the error is large and either diode is conducting as its impedence will be made considerably greater than that of the diodes and capacitors. For small errors, when the diodes are open due either to the signal transient or noise stored on the capacitor, the resistor will provide a very low gain path. This path will allow low level signals

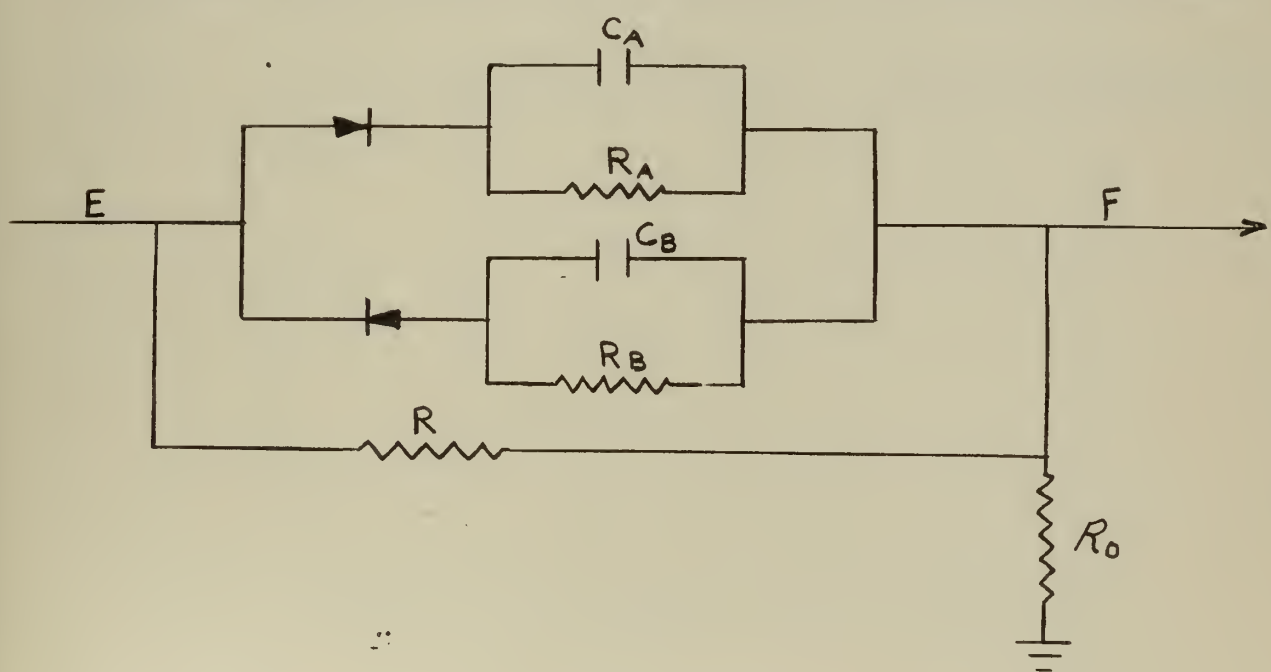


Fig. 7-2 Basic network with modification to reduce noise.

to be transmitted with a reduced gain. This modification was not evaluated experimentally; therefore, only the possibility that it may help alleviate the noise problem, is suggested. Also it may be conjectured that both modifications proposed above might be combined. Caution would be necessary however, as a steady state error might be introduced since the voltage across each of the parallel paths could be equal in magnitude and opposite in polarity for a finite error. This possibility exists for the first modification proposed but is not nearly as likely due to differential in resistor magnitudes. Since this condition would also decrease the added damping effect due to the sign changer during transient, there might well be considerable difficulty in arriving at the proper relationships for compensation.

8.0 SUMMARY AND CONCLUSIONS

8.1 Second Order Compensation

The merit of the nonlinear network for specific applications was well established experimentally. A definite region was found, defined by the damping ratio of the uncompensated system, for which the basic network of Fig. 2-2 significantly improved the system response time. The lower limit for the damping ratio was determined as approximately 0.5 and was only slightly affected by the attenuation ratio used. The region defined by $0.5 \leq \zeta$ has been termed the "Region of Applicability". Data was collected only for damping ratios less than one; however, experimental evidence indicated that there is no upper limit of practical significance. Table 1 compares the uncompensated and compensated system response to a unit step input for different damping ratios. It was verified that the response is identical for any magnitude of step input as long as no other nonlinearities are present in the circuit (Section 4).

Damping Ratio	Normalized Time		Maximum Overshoot (%)	
	Uncompensated	Compensated	Uncompensated	Compensated
0.5	2.2	1.7	17	5
0.6	2.6	1.45	9.5	5
0.7	2.8	1.3	4	5
0.8	3.5	1.2	2	5
0.9	4.0	1.15	1	5
1.0	4.8	1.14	0	5

Table 1. Comparison of response time and peak overshoot for compensated and uncompensated second order system.

For certain applications it may be necessary to design the basic system with a high damping ratio. This often makes the response time excessive.

[illegible]

[illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]

[illegible]

The nonlinear network proposed will decrease the response time as much as 300%, while holding the steady state velocity error coefficient constant. Even greater improvement is possible in some cases if this is not a rigid requirement. If the system gain is decreased, the natural frequency of the original system has effectively decreased with a corresponding increase in damping ratio. As shown previously the nonlinear network increases in effectiveness as damping ratio increases. Fig. 8-1a shows a typical system in the above region. The marked decrease in response time is evident. For those values of damping ratios much lower than 0.5 it was found that the nonlinear network could be used to improve the response, but it was impossible to maintain the overshoot within the 5% limit originally chosen. (See Fig. 8-2a). In addition the design curves (Figs. 6-3, 6-4, and 6-5) show that for this lower range of damping ratios, an ordinary linear lead will give a faster response.

As shown by Fig. 8-1b it was possible to extend the "Region of Applicability" to the lower ζ range by use of the added damping circuit of Section 7. The response was in all cases comparable to, but not faster than, the straight lead. For a d-c system then, it must be assumed that outside the defined region the ordinary d-c lead network would be more satisfactory unless special conditions of noise or other circuit nonlinearities existed. Under such conditions it is conceivable that the modified nonlinear circuit could have significant advantages. It must be concluded that the primary value of the proposed compensator lies in the fact that it will improve the response of the originally sluggish system to such an extent that its response will compare favorably with an originally oscillatory system compensated by a linear lead network.

8.2 Extension to Higher Order Systems

Several third order systems were successfully compensated using both the basic nonlinear and added damping modified circuits. (See Figs. 8-4, 8-5, and 8-6). Results indicated that a second order approximation [9] will give parameters that will allow a designer to eliminate the major part of the trial and error normally required. A typical sluggish third order positioning servo might be represented by an open loop transfer function:

$$G_u = \frac{15.8}{S(S+10)(S+2)}$$

The second order approximation for this system gives a damping ratio of about 0.73 with an oscillation frequency of 1.23 radians per second. Arbitrarily choosing an attenuation factor of 0.1 for the nonlinear compensator, Fig. 6-4 gives $\omega_n \tau_A = 2.25$ from which $\tau_A = 1.84$. An approximate settling time of about 1.1 seconds may also be obtained from the design curve. The actual compensation used for this system on an analog computer simulation was $\tau_A = 1.58$. This value was not critical as 10% change had relatively little effect on the response. The oscillogram of Fig. 8-4 shows the actual results achieved. Note that the settling time is about 1.2 seconds.

It was found that it was possible to compensate third (and higher) order systems which were quite oscillatory with the basic nonlinear circuit. The limit for third order appeared to be when the maximum overshoot of the uncompensated system was approximately equal to that of a second order system with an effective damping ratio of 0.25. However, in all cases where the original overshoot was greater than that of the comparable second order with $\zeta \simeq 0.5$, it was possible to obtain an ordinary lead network which gave as good or better transient response. A typical system

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[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

in this range might be:

$$G_u = \frac{200}{S(S+10)(S+4)}$$

The second order approximation gives a damping ratio of about 0.25 with an oscillation velocity of four radians per second. Extrapolation of the curves of Fig. 6-4 indicates that τ_A must be about 0.11. From Section 4 a response time of about $\frac{1}{\omega_n \zeta}$ or one second may be expected. Fig. 8-5 shows the system as compensated. The actual τ_A was 0.25 and the response time quite close to the estimated one second. From the above it appears that the $\omega_n \tau_A$ curve should be assumed to dip sharply for low damping ratios for extrapolation purposes.

The investigation of third order systems indicates that, as in the case of the second order, the response time of the inherently sluggish system could be considerably improved. The possibility that more highly oscillatory systems of higher order could be compensated was expected from the phase plane analysis. The coast period will not be a straight line but would curve toward correspondence. Several third order systems were compensated with the added damping modification. Satisfactory compensation was obtained both with and without a limitor on the feed forward amplifier. Considerable variation in the circuit parameters was possible due to the added variable of gain in the feed forward path. Figs. 8-4 and 8-6 are typical of the responses obtained, and the same conclusions as for second order are indicated.

Several fourth order systems were examined but no conclusions were possible from experimental results. Compensation was possible for sluggish systems and the results varied for more oscillatory systems depending on the location of the uncompensated system poles. Insufficient data was taken to establish any trends. It did not appear likely that the "Region

of Applicability" could be extended significantly for high order systems, although this possibility does exist.

8.3 Effects of Saturation.

The phase plane is useful for predicting what will occur if saturation is present in the system. For either torque or velocity saturation it may be expected that the maximum velocity for a given magnitude of input will be less than if no saturation were present [14] . It would appear from Fig. 4-6 that the error at maximum velocity will not change as much as the error velocity. Intuitively, it is possible to suggest the general effect of saturation. That is, the response will be slower, with less overshoot. For a high degree of saturation the response will be very similar to that of an overdamped system. The above observation was born out by experimental data. Fig. 8-7 gives a typical response showing the effect of saturation. The particular system shown was designed so that any input of greater than 0.5 radians could be expected to produce some saturation. It is concluded that for the basic network of Fig. 2 the only effect of saturation will be to increase the response time and that the increase will be dependent on the degree of saturation. This conforms with the conclusion of Clegg. [2]

The effects of saturation were considerably more significant with the modified circuit of Fig. 7-1. Results obtained and a possible explanation with a proposed solution were discussed in Section 7.

8.4 Noise and Bandwidth

As in any RC network noise can be expected to be a problem for the nonlinear circuit also. In Section 7 a proposed circuit modification was presented to help alleviate noise difficulties. Experimental observations and intuitive reasoning indicate that high frequency noise will present little

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

problem throughout the "Region of Applicability". The original system being relatively sluggish would not be expected to be very susceptible to high frequency noise. During Mode II the characteristics of the original system are dominant and the error approaches zero during this mode. Furthermore, the presence of the diodes themselves tend to alleviate some noise problems since they effectively isolate the discharge path of the capacitors from the error and output signals.

The circuit modification of Section 7 (Fig. 7-1) with added damping may be less susceptible to noise than a nonlinear lead; however, it seems most likely that the added amplifier will not change the noise characteristics of the basic circuit. Probably its only effect would be to produce a slight 'dither' in the output signal. An evaluation of the possible noise suppression would be desirable since it has been shown that the nonlinear lead with the feed forward modification will provide compensation comparable to a linear lead network. If the noise suppression characteristics are significantly better, its range of possible use with d-c systems may well be extended to those applications where the noise level present precludes use of the d-c lead network.

Bandwidth for a linear system normally gives some indication of rise time; measures in part the ability of the system to reproduce the input signal; and gives an indication of the filtering characteristics of the system. [7] Some bandwidth data was collected and it was found that throughout the "Region of Applicability" the bandwidth for the compensated system was increased approximately 10% over that of the uncompensated system. This is very small compared to the response time improvement achieved. Experimental data indicated a system frequency response for which attenuation was approximately proportional to frequency above the -3db point. At higher

frequencies the attenuation rate was faster but without sharp cutoff.

Several third order systems were checked as to bandwidth for comparison. It was found that the nonlinear network effected the uncompensated system bandwidth only slightly but did eliminate or greatly lower any resonant peaks. From the transient response obtained this was as expected. When the basic network was modified by adding a feed forward path the band width was decreased very slightly. For a linear lead, with parameters adjusted to give satisfactory compensation, the bandwidth was increased.

From the above discussion it appears that it may be assumed that the filtering action of the nonlinear circuit and its ability to reproduce the input signal will be comparable to that of the original uncompensated system. There may be the added advantage of removing any reasonant peak present at low frequencies.

8.5 Conclusions

The proposed network has definite value for limited applications with d-c systems. In any compensating device the specific utility must be determined by the designer with a specific system to improve. The demonstrated ability to improve response time for an inherently sluggish system appears to be the most valuable characteristic. This is possible with the basic network. This network is simple and incorporates inexpensive components. A relatively low attenuation ratio may be chosen, without losing a great deal in transient performance. This will lessen the possibility of system saturation and require a relatively small amount of added gain to hold the velocity error coefficient constant. For a simple positioning servo with low frequency signals the added variable of system gain may well provide the designer with the freedom to make optimum use of the nonlinear network.

The feed forward modification proposed in Section 7 will not improve

[illegible]

the response sufficiently over thatof the basic circuit in the defined "Region of Applicability" to generally recommend its use. The added circuit complexity and stability problems lessen its value in this range. If it could be demonstrated that the modified circuit has better noise characteristics than a linear lead, its value for low damping ratios is apparent. The stability problem for the feed forward circuit probably can be solved by the simple device of limiting the output of the sign changer amplifier.

The response has been shown to be proportional to the input for the otherwise linear circuit. Figure 8-8 shows the effect of introducing a small step input following a larger step. It can be seen that the response to the small step is initially slightly damped, but this is about compensated for by the initial velocity condition resulting from the larger input. Any given compensated system may be expected to give a response without much attenuation as long as the input signal frequency is not much greater than $\frac{1}{6\tau_A}$. This is simply saying that the capacitors must be allowed to discharge sufficiently (3 time constants) in a half period of signal frequency so that the charge upon them for the next signal will not cause excessive attenuation.

The design curves permit the designer to choose suitable parameters for the network quickly. Some adjustment may be necessary, but certainly time will be saved over trial and error methods. For higher order systems it has been shown that second order approximations will give reasonable values for entering the design curves. The settling time may be read from the curves or an approximate value is readily given by $\frac{1}{\zeta\omega_n}$.

Reference 1 gives a detailed discussion of the application of this device to a-c systems. The reader is referred to that work for that particular application.

8.6 Topics for Further Study

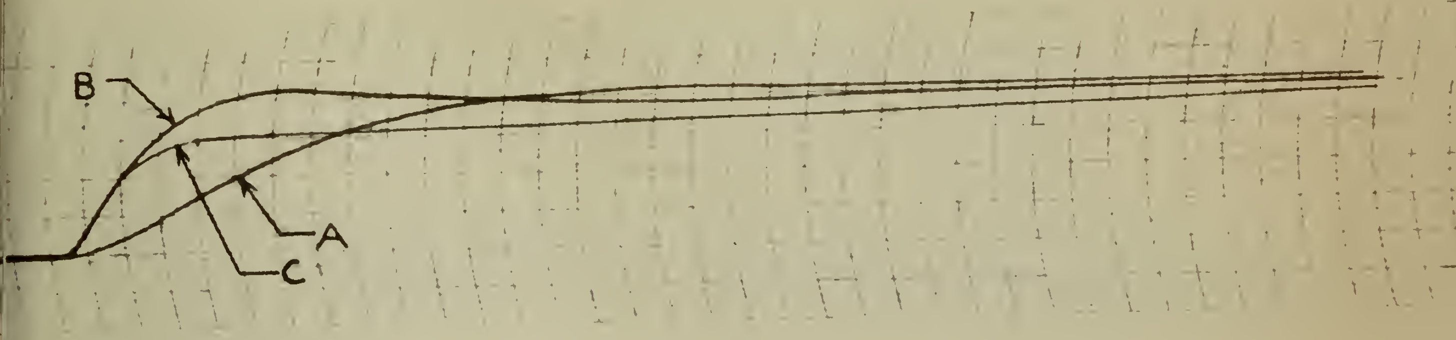
The main objective of this report was to investigate the nonlinear lead network for its application to d-c servo systems and to develop design curves for its use. Several areas for additional study which might be of interest are:

(1) Evaluate the effectiveness of the nonlinear lead network in systems where noise precludes the use of a linear lead compensator.

(2) Investigate the range of application to an uncompensated system which has complex conjugate roots in the forward loop transfer function.

(3) Develop design curves for application of the nonlinear network to a-c carrier systems for frequencies of 60 and 400 cycles.

(4) Compare effectiveness of the nonlinear network to the quasi-optimum relay servo proposed by Thaler. [5,6]

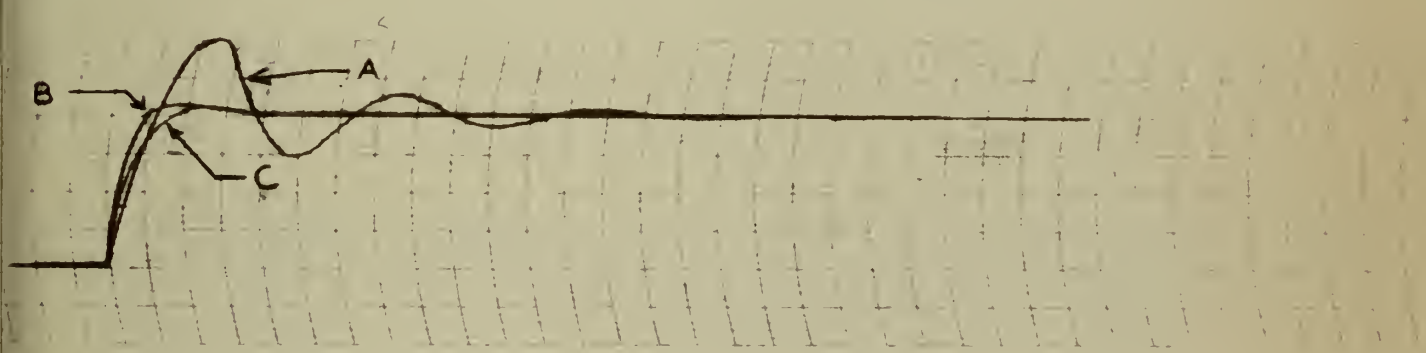


A. Uncompensated

B. Compensated with nonlinear network

C. Linear lead with nonlinear parameters

Fig. 8-1a Second order response to a step for $\zeta = 0.8$

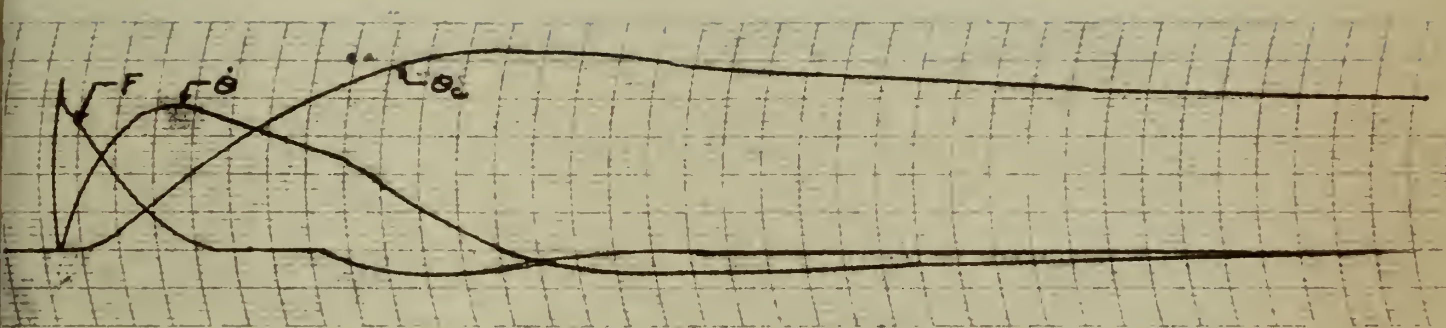


A. Uncompensated

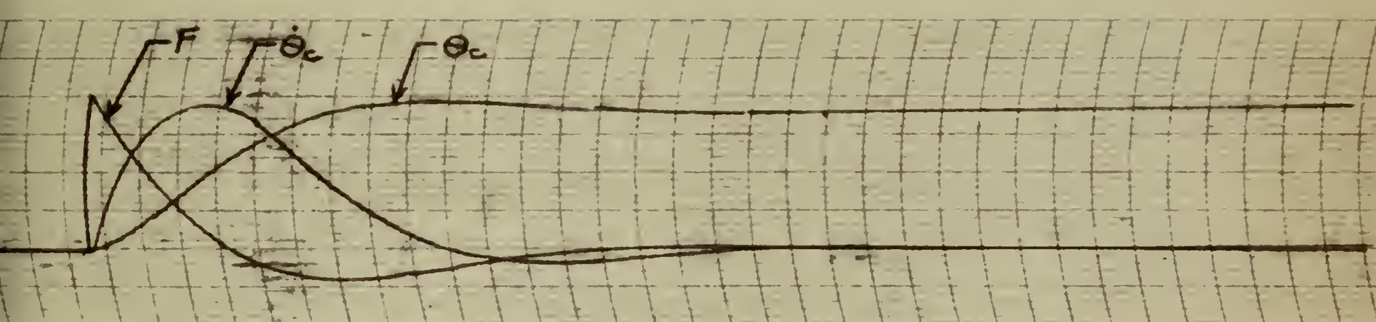
B. Compensated with linear lead

C. Compensated with nonlinear lead network with added damping

Fig. 8-1b Second order response to a step for $\zeta = 0.2$

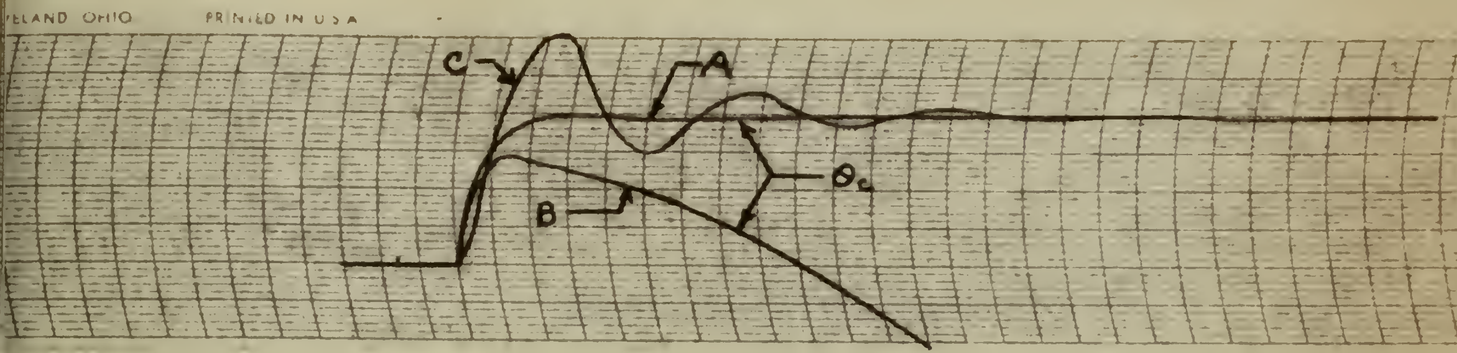


(a) Operation of nonlinear network



(b) Straight lead with identical parameters

Fig. 8-2 Response with linear and nonlinear compensation outside "Region of Applicability"

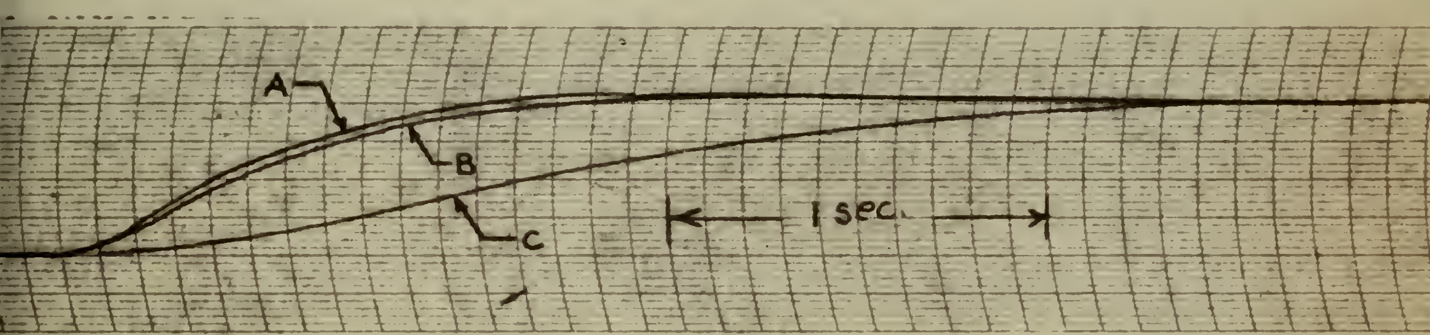


A. Optimum compensation with added damping

B. Instability due to heavy saturation

C. Uncompensated system

Fig. 8-3 Effect of saturation with modified nonlinear compensator



$$G_u = \frac{15.8}{S(S+10)\,(S+2)}$$

A. Compensation with nonlinear network

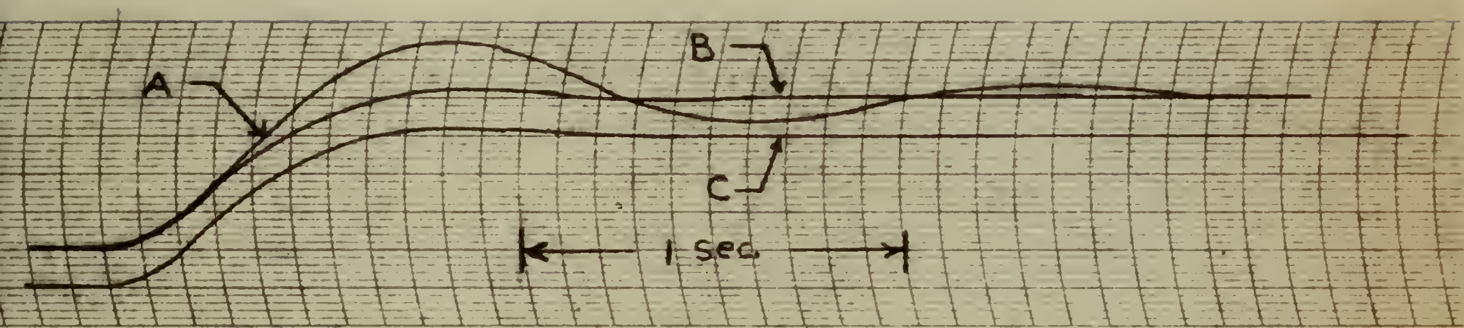
B. Compensation with modified circuit (added damping)

C. Uncompensated system

Fig. 8-4 Response for sluggish third order system

UNDERDAMPED THIRD ORDER

$$G_u = \frac{200}{S(S+10)(S+4)}$$



Time scale: 50mm/sec.

A. Uncompensated

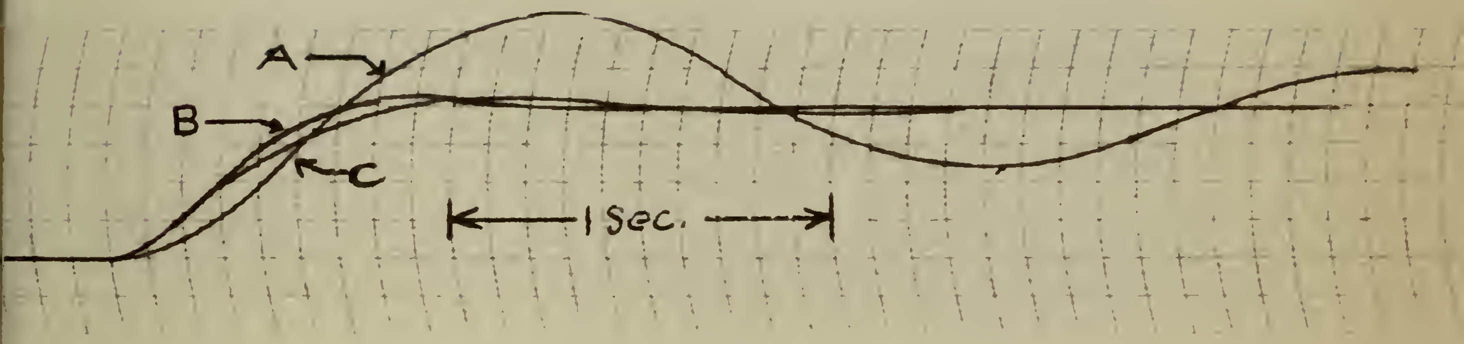
B. Compensated with linear lead

C. Compensated with nonlinear lead

Fig. 8-5 Response for underdamped third order system

[illegible]

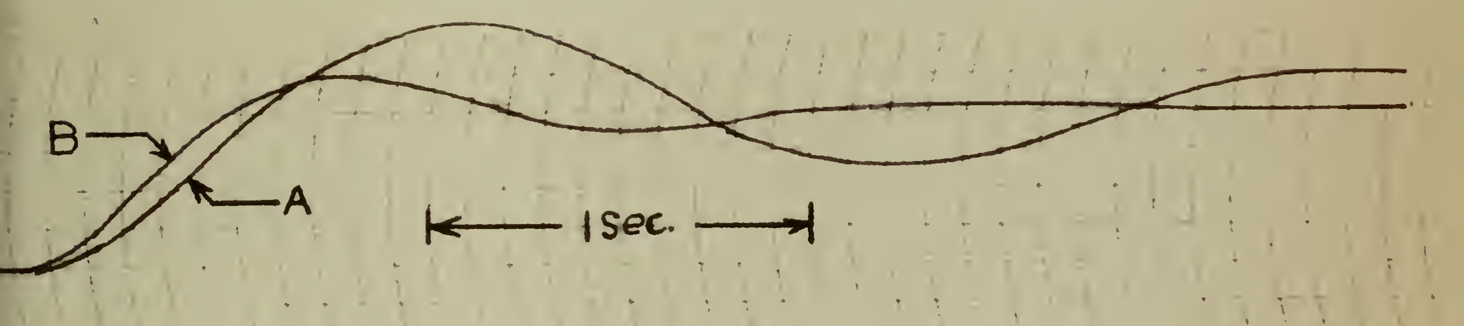
$$G_u = \frac{5}{S(.1S+1)(.6S+1)}$$



A. Uncompensated

B. Compensated with linear lead

C. Compensated with nonlinear lead with added damping

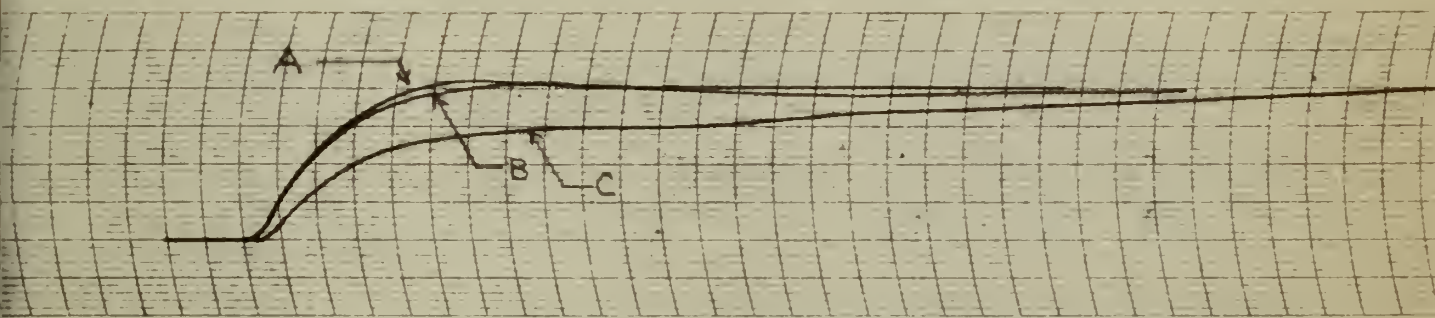


A. Uncompensated

B. Compensated with nonlinear lead

Time scale: [illegible]0 mm/sec.

Fig. [illegible]-6 Comparison of linear and modified nonlinear compensation for oscillatory third order system



$\zeta = 0.8$

A. Step input of 0.25 and 0.5 radian. No saturation

B. Step input of 1.0 radian. Mild saturation

C. Step input of 2.5 radian. Severe saturation

Fig. 8-7 Effect of saturation on second order response

Fig. 8-8 Response for step superimposed on step input

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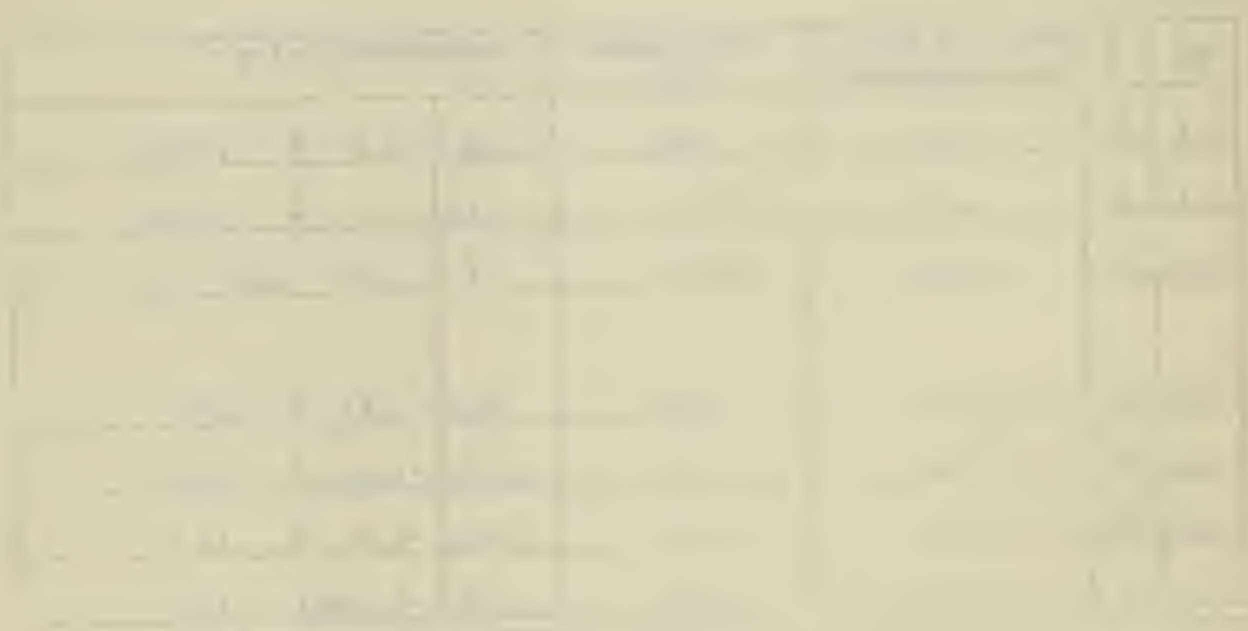
APPENDIX I - Pole and Zero Location for Satisfactory Compensation.

In Sec. III several limitations for placement of the compensator open loop zero are derived. The trends discussed in that section were verified by the actual compensation required, as determined by analog computer. The below tabulation of open and closed loop poles is included to demonstrate the placement required to achieve the desired compensation. The information is available from the design curves, but only the normalized zero ($\frac{1}{\omega_n \tau_A}$) can be obtained without making use of the mathematical relations of Sec. III. All values listed are normalized so that for a real system they are increased in direct proportion to the natural frequency. The closed loop poles are only approximate solutions for the characteristic equation (Eq. 27).

α	ζ	Open Loop Pole (Uncompensated)	Compensator Zero	Compensator Poles	
.1	.8	1.6	.382	.26	$-2.6 \pm j2.85$
.1	.6	1.2	0.60	.415	$-3.4 \pm j1.68$
.1	.5	1.0	0.89	.73	-1.74 and -7.41
.2	.8	1.6	.324	.243	$-1.65 \pm j2.0$
.2	.6	1.2	.641	.452	$-1.97 \pm j1.76$
.05	.8	1.6	.419	.214	$-4.9 \pm j2.75$
.05	.6	1.2	.625	.427	-2.8 and -10.7

Normalized values of Mode I poles and zeros required to obtain 5% overshoot to a step input.

It is apparent that as the damping ratio is decreased the zero moves from near the orgin ($\frac{1}{\tau_A} \to 0$) toward the uncompensated system pole (2 ζ). It is seen that the normalized zeros for all three values of attenuation ratio are quite close together (as can be observed from overlaying the three design curves). The fact that some of the closed loop



poles are complex and others all real has no apparent significance. This is as expected since the only requirement is that Mode I terminate at the proper instant.

APPENDIX II - Sample Computations

An analysis of applied compensation will be made using the developed mathematical relations and checking with experimental data. Consider a basic second order system that may be described by the following transfer function and written in the form of Eq. 1:

$$G_u = \frac{(4.17)^2}{S(S+5)} = \frac{17.36}{S(S+5)} \tag{1A}$$

The above servo system would be rather sluggish but have about 9% maximum overshoot. The rise time to the peak would be approximately 0.9 sec. The velocity coefficient is $\frac{17.36}{5}$ = 3.47 and this is assumed to be the minimum allowable value. For the most rapid rise time an $\alpha = 0.05$ is chosen. From the proper design curve (Fig. 6-5) $\omega_n \tau_A$ may be obtained by entering with the system equivalent ζ which for Eq. 1A is 0.6. A value of about 1.6 is obtained for $\omega_n \tau_A$ from which:

$$\tau_A = \frac{1.6}{4.17} = 0.385 \tag{2A}$$

$$\frac{1}{\tau_A} = 2.6$$

From Eq. 2, Section 3.1:

$$G_L = K\left[\frac{S + \frac{1}{\tau_A}}{S + \frac{K}{\tau_A}}\right] = 20\left(\frac{S + 2.6}{S + 52}\right)$$

From Eqs. 3a and 4a:

$$G_o G_L = G_o = \frac{347.2(S+2.6)}{S(S+5)(S+52)}$$

$$G_c = \frac{347(S+2.6)}{S^3 + 57S^2 + 607.2S + 902.7} \tag{3A}$$

Eq. 3A is then the basic equation describing Mode I operation. The first operation required is to obtain the roots of the characteristic equation. By synethetic division one real root was obtained and the cubic equation factored:

$$S^3 + 57S^2 + 607.2S + 902.7 \simeq (S+1.78)(S+11.7)(S+43.5) \tag{4A}$$

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$$[illegible]$$

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The problem is considerably simplified due to the fact that the roots do not include a complex conjugate pair. This is not to be construed to be true in general; indeed, the characteristic equation will normally contain a complex conjugate pair.

Assuming a step input of magnitude R and using Eq. 4A and referring to Eqs. 6 and 7 Section 3:

$$\theta_c = \frac{347.2\,(s+2.6)\,R}{s(s+1.78)(s+11.7)(s+43.5)} \qquad (5A)$$

$$F = \frac{20\,(s+5)(s+2.6)\,R}{s(s+1.78)(s+11.7)(s+43.5)} \qquad (6A)$$

It is now necessary to obtain the inverse transform for F, set it equal to zero and solve for the cutoff time (t_{c_1}). Although not difficult, this would be a tedious task if many problems had to be worked for trial and error compensation. If the order of G_u were increased the problem would be nearly insurmountable and most certainly not practical. The approximate inverse transforms were obtained from Eqs. 5A and 6A:

$$\theta_c = R\left[1 - .388\,\epsilon^{-1.78t} + 0.243\,\epsilon^{-43.5 8} - 0.855\,\epsilon^{-11.7t}\right]$$

$$F = 20R\left[.00644\,\epsilon^{-1.78t} - .191\,\epsilon^{-11.7t} + 1.192\,\epsilon^{-43.5t}\right]$$

By differentiation:

$$\dot{\theta}_c \simeq R\left[0.69^{-1.78t} + 9.88\epsilon^{-11.7t} - 10.57\,\epsilon^{-43.5t}\right]$$

By trial and error the time for which F first goes to zero was found to be about 0.06 sec. Position and velocity are then evaluated:

$$\theta_{o1} = \theta_c(t_{c_1}) = \theta_c(0.06) \qquad = .25R \qquad \text{rad}$$

$$\dot{\theta}_{o1} = \dot{\theta}_c(t_{c_1}) = \dot{\theta}_c(0.06) \qquad = 4.70R \qquad \text{rad/sec}$$

$$E_{o1} = \theta_R - \theta_c = R(1-0.25) \qquad = 0.75R$$

Note that $\left|\frac{\dot{\theta}_{c1}}{E_{o1}}\right| = \frac{.47}{.75} \simeq 6.25 > 2\zeta\omega_n = 5$ indicating that the system will coast in Mode II and there will be some overshoot, the amount determined primarily by the Mode III braking forces present. From Mode II Eqs. 9

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through 10a:

$$t_{c2} = \frac{1}{5} \ln \left(\frac{4.7}{4.7 - 5(.75)} \right) = \frac{1}{5} \ln 4.95 \simeq 0.32 \text{ sec}$$

$$\dot{\Theta}(t_{c2}) = 4.7 R \epsilon^{-5t_{c2}} = 0.95 R = \dot{\Theta}_{12}$$

$$\ddot{\Theta}(t_{c2}) = -5 \dot{\Theta}_{12} = -4.75 R = \ddot{\Theta}_{12}$$

The conditions for evaluation of Mode III are now available.

From Eq. 13:

$$|E| = \frac{0.95 R (S + 52)}{(S + 1.78)(S + 11.7)(S + 43.5)}$$

Transforming and differentiating:

$$|E| = .95R\left[.121 \epsilon^{-1.78t} - .128 \epsilon^{-11.7t} + .0064 \epsilon^{-43.5t}\right]$$

$$|\dot{E}| = R\left[1.41 \epsilon^{-11.7t} - .205 \epsilon^{-1.78t} - .255 \epsilon^{-43.5t}\right]$$

It is determined that error velocity becomes zero at about 0.19 seconds

thus: $\Theta_c(0.19) = \Theta(t_3) = 0.06 R$

Since t_3 occurs at the instant of maximum overshoot the maximum percentage of overshoot for the compensated system is determined as 6%. The rise time is $t_{c1} + t_{c2} \simeq 0.38$ sec. The oscillogram of Θ vs. F shows the comparison between mathematical results. The computed and experimental values for cutoff time t_c, and the rise time correspond very closely. The computer shows an overshoot slightly less than 5% compared to a computed value of 6%. The value of Θ_{c1}, from the computer tape indicated about 0.29R compared to the 0.25R computed. The correlation is better than would normally be expected because of minor experimental inaccuracies and mathematical approximations. If an α of 0.1 had been chosen for the compensation the value of $\frac{1}{T_A}$ would have been 2.5, a very slight change from that for α equal to 0.05. However, the characteristic equation given by Eq. 5 becomes:

$$D = S^3 + 30S^2 + 423.6S + 434$$

Which factors to approximately $(S+1.1)(S^2+29.5S+392)$ which will of course lead to complex conjugate roots. Then it would be more difficult to go through the preceding analysis. The nature of the roots during Mode I operation then can be seen to depend largely on the value of α and are not in themselves greatly significant.

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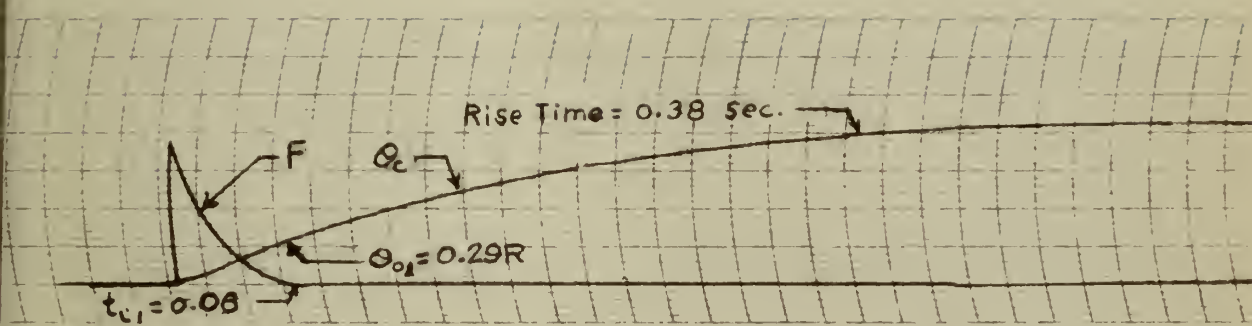

Fig. A -1 Experimental verification of calculated response